Competitive Analysis of Online Price Discount Replacement Problem

Lili Ding, Xinmin Liu, Wanglin Kang
College of Economics and Management
Shandong University of Science and Technology
Shandong, 266510 P.R. China
llding@mail.xjtu.edu.cn

Abstract

When a paid price discount activity occurs, the decision-maker must decide whether or not and when to pay the additional fees for preferential price in an online fashion. This problem which generalizes the basic leasing problem has been introduced and studied by Fleischer et al. In this paper, we extend the basic model to consider different price discount replacement case and present the optimal deterministic online CA algorithm. Moreover, the risk-reward framework is introduced into the model to allow the decision-makers to manage their risk and utilize their forecasts. We give a new online risk algorithm and derive an improved competitive ratio, which is the function of the risk tolerance λ and forecast F. It is found that with the competitive risk analysis, the flexible competitive ratio has the decreasing character of the risk tolerance.

1. Introduction

Price discounts are commonly used for selling goods in the economic market. Some favorable prices are free, and the others are paid. For example, VIP member and discount card, which cost C and entitle their holders to β price reduction of goods for the time of T. For a decision-maker, he usually does not know how many he will buy, which is a typical online problem. This problem, called the Bahncard problem, was proposed by Fleischer [4] in the computer science field. In his paper two optimal deterministic online algorithms which achieved an optimal competitive ratio were present. Fujiwara et al. [5] integrated a possibility distribution assumption into the traditional competitive analysis, where they assume that the input sequence were subject to an exponential distribution. Ding et al. [3] pointed out that the decision-makers were often confronted with the interest rate, which may be an essential feature of any reasonable economic models and gave the optimal deterministic online algorithm. The above Bahncard problem has various interesting applications. In all of these applications the basic question is when to switch from one activity to another more rewarding one. For example, when $\beta = 0$ and the rental is equal, the Bahncard problem is of course precisely *leasing problem* [9] . If the discount rate becomes the weight of packets, then this problem also can be considered as the *TCP problem* [2] .

A systematic study of online algorithms was given by Sleator and Tarjan [8], who compared the performance of an online algorithm with that of an optimal offline algorithm. Karlin et al. [6] introduced the notion of a competitive ratio. Note that the use of the competitive ratio for the evaluation of online algorithm is called competitive analysis. An online algorithm is said to be r-competitive ($r \geq 1$), if, given any instance of the problem denoted by σ, the cost of the solution given by the online algorithm is no more than r multiplied by that of an optimal offline algorithm: $Cost_{online}(\sigma) \leq r \cdot Cost_{optimal}(\sigma)$. The optimal competitive ratio for an online problem is $r^* = inf_{online} r(online)$.

In this paper, we extend the original fixed discount rate to consider two kinds of price discount activities. As an online decision-maker, one has to decide whether and when to replace these two price discount activities. We show that never-buy algorithm, applied by most decision-makers, is not better than immediately-buy algorithm. Then an optimal online deterministic algorithm based on the traditional competitive analysis is present. Furthermore, we give a new online risk algorithm and derive the improved competitive ratio (with respect to the ratio for the traditional online problem), which is the function of the risk tolerance λ and forecast F. This simple online risk algorithm provides a smooth generalization of CA algorithm, which is the special case obtained with $\lambda = 1$.

2. Two-kind price discount replacement

In this paper we propose a two-kind price discount replacement model. For example, the Swiss Federal Railways offers two kinds of Half-Fare cards (cost CHF 150 and

CHF 250, respectively) with different discount rates to attract more travelers. Suppose that the online decision-maker wishes to pay a new discount activity D_2 with a greater discount of β_2, after he has hold a discount right of D_1 with a smaller discount rate β_1. However, the greater discount rate, the more cost. Assume that there is a request sequence, $\sigma = (\sigma_1, \sigma_2, \ldots \sigma_n)$, presented in a chronological order. Each $\sigma_i = (t_i, p_i)$, where $t_i \geq 0$ is the time and $p_i \geq 0$ is the regular price of the goods, and we have that $0 \leq t_1 < t_2 < \cdots$. At t_i, if the online decision-maker has paid the additional fee for the favorable price of goods, then σ_i is called the reduced request, otherwise, σ_i is called the regular request. We give an online algorithm that is $2 - \frac{\beta_2}{\beta_1}$-competitive and we also prove that this result is best-possible for two-kind price discount replacement structure.

2.1. The lower bound

In reality, there is a common algorithm for the decision-makers, which is never to pay the second D_2. However, such never-buy algorithm is $\frac{\beta_1}{\beta_2}$−competitive. Before we analyze the other online algorithms, we show a lower bound on the deterministic competitive ratio. The result is obtained by considering the optimal offline algorithm as an vicious adversary. Namely, regardless of what strategy the online decision-maker chooses, the adversary can specify the request sequence in a way that the online algorithm performs badly.

Theorem 1. *No deterministic algorithm for the two-kind price discount replacement problem can obtain a better competitive ratio than* $2 - \beta_2$.

Proof: Let ALG be an online algorithm for this problem. We will prove that ALG will not achieve a competitive ratio less than r. Suppose that an adversary will construct a special request sequence consisting of the following rules(see in Figure 1): (1) The cost of each request is an arbitrarily small constant ε, so that all requests are in intervals of length T. (2) If the online decision-maker does not pay D_2, the adversary continues the game. (3) Once the online decision-maker pays D_2, the adversary will stop showing requests. Let s_1 be the accumulative regular cost after paying the first D_1. We get the critical cost $s^* = \frac{c_2}{\beta_1 - \beta_2}$ by solving the equation $c_1 + \beta_1(s_1 + s_2) = c_1 + \beta_1 s_1 + c_2 + \beta_2 s_2$. Note that if ALG wants to be better than $\frac{\beta_1}{\beta_2}$− competitive, it must eventually pay D_2. Without loss of generality, assume that ALG does not pay the second D_2 until the total regular cost after its appearance is up to $\tilde{s}$, not including the current request. Then the adversary stops showing requests. We discuss the following two cases.

Case 1. $\tilde{s} + \varepsilon \leq s^*$. The online decision-maker will pay the second D_2 but the adversary does not pay it. Therefore the competitive ratio is

$$r^1 = \frac{c_1 + \beta_1 s_1 + \beta_1 \tilde{s} + c_2 + \beta_2 \varepsilon}{c_1 + \beta_1(s_1 + \tilde{s} + \varepsilon)} \tag{1}$$

Note that $\frac{\partial r^1}{\partial \tilde{s}} < 0$. r^1 has the monotonic decreasing character of $\tilde{s}$. Therefore, the online decision-maker will take the maximum possible at $\tilde{s} = s^* - \varepsilon$. Set $c_1 + \beta_1 s_1 = \eta$. From (1) the following inequality can be achieved

$$r^1 \geq 1 + \frac{(1-\beta_2)c_2 - (1-\beta_2)^2\varepsilon}{\eta(1-\beta_2) + c_2} \tag{2}$$

Case 2. $\tilde{s} + \varepsilon > s^*$. Both the adversary and online decision-maker will pay the second D_2. The competitive ratio is

$$r^2 = \frac{c_1 + \beta_1 s_1 + \beta_1 \tilde{s} + c_2 + \beta_2 \varepsilon}{c_1 + \beta_1 s_1 + c_2 + \beta_2(\tilde{s} + \varepsilon)} \tag{3}$$

From (3), $\frac{\partial r^2}{\partial \tilde{s}} > 0$. We know that r^2 is the increasing function of $\tilde{s}$, and then the adversary will get the minimum of $s^* - \varepsilon$. We can achieve the following result.

$$r^2 > 1 + \frac{(1-\beta_2)c_2 - (1-\beta_2)^2\varepsilon}{\eta(1-\beta_2) + c_2} \tag{4}$$

From cases 1 and 2, we get a lower bound of $r = min\{r^1, r^2\}$. When $\varepsilon \to 0$ and $\eta \to 0$, we obtain the following result.

$$r > 2 - \beta_2 \tag{5}$$

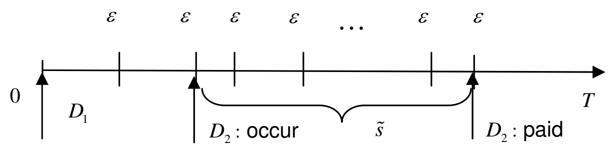

Figure 1. The special request sequence by the adversary.

2.2. The upper bound

The online decision-makers, sometimes, would like to pay D_2 immediately only if it appears, denoted by the immediately-buy algorithm. It is evident that the competitive ratio of the immediately-buy algorithm is c_2. These algorithms can not safeguard against a sequence of several expensive requests or cheap ones, respectively. Therefore, we consider the following online accumulative algorithm(CA):
(1) If $s_2 < s^*$, then never pay D_2.
(2) Otherwise, pay D_2 until the accumulative regular cost is up to s^* after the appearance of D_2.

Theorem 2. *CA is* $1 + \frac{(\beta_1-\beta_2)c_2}{\eta(\beta_1-\beta_2)+\beta_1 c_2}$ *−competitive for the two-kind price discount replacement problem.*

Proof: Suppose that the second D_2 is available at time t. Let s_2 be the total regular cost after t. Let ALG be the online algorithm that the decision-maker does not pay it until the total regular cost after t is equal to $\tilde{s}$. Then, ALG pays $\eta + \beta_1 \tilde{s} + c_2 + \beta_2(s_2 - \tilde{s})$. The offline decision maker has the optimal choice to pay the second D_2 immediately after it is available, otherwise never. Thus, the cost incurred by the offline decision-maker is $min\{\eta + \beta_1 s_2, \eta + c_2 + \beta_2 s_2\}$

If $s_2 < s^*$, the online and offline decision-makers incur the same cost and achieve the competitive ratio of 1. On the other hand, if $s_2 \geq s^*$, then the competitive ratio is

$$r = \frac{\eta + \beta_1 \tilde{s} + c_2 + \beta_2(s_2 - \tilde{s})}{\eta + c_2 + \beta_2 s_2} \tag{6}$$

According to the competitive analysis, the offline decision-maker will choose $s_2 = \tilde{s}$, enforcing the larger ratio of r. We can get derivative $\frac{\partial r}{\partial \tilde{s}} > 0$. It follows that r is an increasing function of $\tilde{s}$. Therefore, the best attainable ratio is obtained by setting $\tilde{s} = s^*$. Thus, we obtain

$$r \leq \frac{\eta + \beta_1 \tilde{s} + c_2}{\eta + c_2 + \beta_2 \tilde{s}} = 1 + \frac{(\beta_1 - \beta_2)c_2}{\eta(\beta_1 - \beta_2) + \beta_1 c_2} \tag{7}$$

Hence the online decision-maker chooses $\tilde{s} = s^*$ and the best attainable competitive ratio of r^*_{CA} is achieved by the optimal CA algorithm. Namely, if $\tilde{s}$ at some time is at least s^*, then the decision-maker pays the second D_2; otherwise, never pay it.

Corollary 1. The optimal competitive ratio obtained is $2 - \frac{\beta_2}{\beta_1}$, when $\eta \to 0$.

Proof: When $\eta \to 0$, substituting η into (7) and the optimal competitive ratio is at most $2 - \frac{\beta_2}{\beta_1}$. Our analysis provides a smooth generalization of Fleischer' results, which are the special cases obtained with $\beta_1 = 1$.

3. Competitive risk analysis

The above competitive analysis is the most fundamental and significant approach, yet it has been criticized as making too conservative assumption about future input sequences. Especially in the economic issues, many decision-makers do not seek to minimize risk, but to manage it. MacCrimmon and Wehrung [7] introduce a basic risk paradigm as the basis for studying risk. Al-Binali [1] takes a risk by assuming that input sequence will obey some constraints. We provide the following risk-reward framework displayed in their manners. Given any online deterministic algorithm ALG, define the risk of ALG to be $Risk(ALG) = r_{ALG}/r^*$. It is clear that the risk of any online algorithm is ≥ 1 and the lower the risk is, the better its performance guarantees. Next, define a forecast denoted by F as any subset of the allowable input sequence. The online decision-maker specifies a risk tolerance λ. This means that the decision-maker is willing to use the restricted algorithms in $\xi = \{ALG : Risk(ALG) \leq \lambda\}$. Each of the algorithms in ξ thus has a competitive ratio of at most λr^*. Fix any forecast F. An optimal risk algorithm, according to this risk and reward framework, is an algorithm from ξ that minimizes the competitive ratio, restricted to input sequences from F. Formally, the restricted competitive ratio $\hat{r}$ of any risk algorithm can be parameterized by the constrains of the total input sequences from F such that

$$\hat{r}_{ALG} = sup_{\sigma \in F}\{Cost_{ALG}(\sigma)/Cost_{OPT}(\sigma)\} \tag{8}$$

Thus the optimal restricted competitive ratio by ALG^* with respect to a forecast F can be achieved from $\hat{r}^* = inf_{ALG}\{\hat{r}_{ALG} : ALG \in \xi\}$. The reward of the optimal risk algorithm ALG^* denoted by g_{ALG} is measured by the ratio of the optimal competitive ratio to the restricted ratio. The optimal risk algorithm ALG^* with respect to a forecast F satisfies

$$g_{ALG^*} = sup_{ALG}\{r^*/\hat{r}_{ALG} : ALG \in \xi\} \tag{9}$$

3.1. The online risk algorithm

We analyze two-kind price discount replacement problem in the risk-reward framework based on the two possible forecasts of $s_2 < s^*$ and $s_2 \geq s^*$. For the case of $s_2 < s^*$, the optimal competitive ratio is 1. It is because that both the offline and online decision-makers will never pay the second D_2 with this forecast. For the case of $s_2 \geq s^*$, we present the following risk algorithm:

Risk Tolerance Algorithm (RTA):

Given η, λ, a new D_2 and a forecast of $s_2 \geq s^*$, the online decision-maker would not pay D_2 until the total regular cost of $\tilde{s}$ is up to $\frac{(c_2+\eta)(\lambda r^*_{CA}-1)}{\beta_1-\beta_2\lambda r^*_{CA}}$, otherwise, never pay D_2.

The risk tolerance algorithm is present to analyze such complex situation in two stages. In first stage the risk algorithm is under the threat that forecast is incorrect, and chooses some threshold to ensure a competitive ratio of $\hat{r} \leq \lambda r^*$. The second stage begins when the forecast comes true, the risk algorithm chooses a threshold as small as possible subject to the constraints of the stage 1.

3.2. Optimality and risk tolerance

The following result shows that the introduction of forecast improves the competitive analysis performance of the accumulative algorithm, if $s_2 \geq s^*$ is correct.

Theorem 3. *If the forecast of* $s_2 \geq s^*$ *is correct, the optimal restricted competitive ratio is*

$$1 + \frac{(\beta_1 - \beta_2)(c_2 - (\lambda r^*_{CA} - 1)\eta)}{\eta(\lambda r^*_{CA} - 1)(\beta_1 - \beta_2) + c_2((\lambda r^*_{CA} - 1)\beta_1 + \beta_2)}.$$

Proof: If $s_2 \geq s^*$ is correct, then the online decision-maker would choose an optimal risk algorithm from ξ to obtain the more reward based on his tolerance. The offline decision-maker would pay the second D_2 as soon as it appears. Therefore the restricted competitive ratio of the risk algorithm is

$$\hat{r}_{RTA} = \frac{\eta + \beta_1 \tilde{s} + c_2 + \beta_2(s_2 - \tilde{s})}{\eta + c_2 + \beta_2 s_2} \quad (10)$$

Since we want to minimize $\hat{r}_{RTA}$, we want s_2 as small as possible subject to $\hat{r}_{RTA} \leq \lambda r^*_{CA}$ from the following two cases.

Case 1. $s_2 < s^*$. According to the preceding risk definition, the online decision-maker can obtain the inequality $\frac{\eta+\beta_1\tilde{s}+c_2+\beta_2(s_2-\tilde{s})}{\eta+\beta_1 s_2} \leq \lambda r^*_{CA}$. If $c_2 - (\lambda r^*_{CA} - 1)\eta > 0$, then we obtain

$$\tilde{s} \geq \frac{c_2 - (\lambda r^*_{CA} - 1)\eta}{(\lambda r^*_{CA} - 1)\beta_1} \quad (11)$$

Case 2. $s_2 \geq s^*$. The restricted competitive ratio in the false forecast satisfies $\frac{\eta+\beta_1\tilde{s}+c_2+\beta_2(s_2-\tilde{s})}{\eta+c_2+\beta_2 s_2} \leq \lambda r^*_{CA}$ from the above competitive risk analysis framework. Thus we get the following result.

$$\tilde{s} \leq \frac{(c_2 + \eta)(\lambda r^*_{CA} - 1)}{\beta_1 - \beta_2 \lambda r^*_{CA}} \quad (12)$$

For the online decision-maker, he knows the function of $\hat{r}_{RTA}$ in case 2 is monotonic increasing with $\tilde{s}$. The smaller s_2, the less $\hat{r}_{RTA}$. Therefore substituting $\frac{c_2-(\lambda r^*_{CA}-1)\eta}{(\lambda r^*_{CA}-1)\beta_1}$ into the function of restricted competitive ratio of (10), we obtain the restricted optimal competitive ratio

$$1 + \frac{(\beta_1 - \beta_2)(c_2 - (\lambda r^*_{CA} - 1)\eta)}{\eta(\lambda r^*_{CA} - 1)(\beta_1 - \beta_2) + c_2((\lambda r^*_{CA} - 1)\beta_1 + \beta_2)}.$$

Corollary 2. The optimal restricted competitive ratio is at most $1 + \frac{\beta_1 - \beta_2}{(\lambda r^*_{CA}-1)\beta_1+\beta_2}$ for $\eta \to 0$.

Corollary 3. When $\eta \to 0$ and $\lambda = 1$, the optimal restricted competitive ratio is $2 - \frac{\beta_2}{\beta_1}$.

Proof: Substitute $\eta \to 0$ and $\lambda = 1$ into the function of restricted competitive ratio and achieve the corollary. The risk tolerance algorithm generalizes the accumulative algorithm as a special case when $\lambda = 1$.

Corollary 4. If the forecast of $s_2 \geq s^*$ is correct, the optimal reward is $g_{RTA} = \frac{r^*_{CA}}{\hat{r}_{RTA}}$.

Proof: $g_{RTA} = \frac{r^*_{CA}}{\hat{r}_{RTA}} = \frac{(\lambda r^*_{CA}-1)\beta_1+\beta_2}{\lambda\beta_1}$, when $\eta \to 0$.

4. Conclusions

The classical competitive analysis is the most fundamental and important approach to study online problems. But it is not very flexible since it avoids making assumptions about future inputs. In this paper, we provide a risk-reward framework, which allows the decision-makers to manage their risk and utilize their forecasts. But how to improve the performance of the competitive algorithm by other methods, such as probability statistics, is a direction. Another direction is the competitive analysis about more discount activities, for example three or more discount rates.

Acknowledgements

This work is supported by Soft Science of Shandong Province(No.2008RKB123); Natural Science Foundation of Shandong Province(No.Z2007H04).

References

[1] S. Al-Binali. A risk- reward framework for the competitive analysis of financial games. *Algorithmica*, 25:99–115, 1999.

[2] S. Albers and H. Bals. Dynamic tcp acknowledgement: Penalizing long delays. *In: Proc. of 14th ACM-SIAM Symposium on Discrete Algorithms*, pages 47–55, 2003.

[3] L. L. Ding, Y. F. Xu, and S. H. Hu. The bahncard problem with interest rate and risk. *In: Proc. of International Workshop on Internet and Network Economics*, pages 307–314, 2005.

[4] R. Fleischer. On the bahncard problem. *Theoretical Computer Science*, 268:161–174, 2001.

[5] H. Fujiwara and K. Iwama. Average-case competitive analysis for ski-rental problems. *Algorithmica*, 42:95–107, 2005.

[6] A. Karlin, M. Manasse, L. Rudolph, and D. Sleator. Competitive snoopy caching. *Algorithmica*, 3:79–119, 1988.

[7] K. R. MacCrimmon, D. A. Wehrung, and W. T. Stanbury. Taking risks: the management of uncertainty. *The Journal of Risk and Insurance*, 54:391–394, 1987.

[8] D. Sleator and R. Tarjan. Amortized efficiency of list update and paging rules. *Communications of the ACM*, 28(2):202–208, 1985.

[9] Y. F. Xu, W. J. Xu, and H. Y. Li. On the on-line rent-or-buy problem in probabilistic environments. *Journal of Global Optimization*, 38:1–20, 2007.