B-invexity and B-monotonicity of Non-differentiable Functions

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Abstract. Several new kinds of generalized B-invexity and generalized invariant B-monotonicity are introduced for non-differentiable functions. The relations among (quasi) B-preinvexity, (pseudo, quasi) B-invexity and invariant (pseudo, quasi) B-monotonicity are studied by using the Clarke’s subdifferential of non-differentiable functions and a series of examples. Some new results are obtained, which can be viewed as an extension of some known results.

Keywords: B-preinvexity, B-invexity, invariant B-monotonicity, Clarke’s subdifferential, relations.

1 Introduction

Convexity is a common assumption made in mathematical programming. There have been increasing attempts to weaken the convexity of objective functions, see for example [1-11] and references therein. An interested generalization for convexity is B-vexity, which was introduced and studied by Bector and Singh [4]. They studied some properties of B-vex functions in the settings of differentiable and non-differentiable, respectively. Later, B-preinvexity was introduced by Suneja et al. [5] as an extension of preinvexity and B-vexity. At the same time, B-vexity was generalized to pseudo (quasi) B-vexity and (pseudo, quasi) B-invexity in the setting of differential in [6]. Recently, B-vexity was studied by Li et al. [7] in the setting of non-differential and some necessary and sufficient results are obtained by means of the Clarke’s subdifferential.

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A concept related to the convexity is the monotonicity of the mappings. In 1990, Karamardian and Schaible [8] studied the relations between the convexity of a real-valued function and the monotonicity of its gradient mapping. Yang et al. [9] and Jabarootian and Zafarani [10] investigated the relations between invexity and generalized invariant monotonicity in the settings of differentiable and non-differentiable, respectively.

Motivated and inspired by works in [7-9], in this paper, we will introduce several new notions of generalized B-invexity and generalized invariant B-monotonicity, which are called pseudo (quasi) B-invexity and invariant pseudo (quasi) B-monotonicity, and study the relations among (quasi) B-preinvexity, (pseudo, quasi) B-invexity and invariant (pseudo, quasi) B-monotonicity by means of the Clarke’s subdifferential of non-differentiable functions and a series of examples. Some new results are obtained, which can be viewed as an extension and improvement of corresponding results in [2,6,7,10].

2 Generalized B-invexity and Generalized Invariant B-monotonicity

Throughout this paper, let $X$ be a real Banach space endowed with a norm $\| \cdot \|$ and dual space $X^*$. We denote by $2^X$, $\{ \cdot \}$, $[x^1, x^2]$ and $(x^1, x^2)$ the family of all nonempty subset of $X^*$, the dual pair between $X$ and $X^*$, the line segment for $x^1, x^2 \in X$ and the interior of $[x^1, x^2]$, respectively. Let $K$ be a nonempty subset of $X$, $\eta : K \times K \to X$ a vector valued mapping and $f : X \to R$ a function.

$K$ is said to be an invex set with respect to $\eta$ (see [1]) if for any $x^1, x^2 \in K$ and any $\lambda \in [0,1]$ one has $x^1 + \lambda \eta(x^2, x^1) \in K$.

From now on, unless otherwise specified, we assume that $K$ is a nonempty invex set with respect to $\eta$.

Let $f$ be locally Lipschitz continuous at $x \in X$ and $v$ be an any other vector in $X$. The Clarke’s generalized directional derivative of $f$ at $x$ in the direction $v$ is defined by $f^g(x; v) = \limsup_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}$. The Clarke’s generalized subdifferential of $f$ at $x$ is defined by $\partial^c f(x) = \{ \xi \in X^* : f^g(x; v) \geq \langle \xi, v \rangle, \quad \forall v \in X \}$. As shown in [11], $\partial^c f(x)$ is a nonempty convex set and $f^g(x; v) = \max_{\xi \in \partial^c f(x)} \langle \xi, v \rangle$ for all $v \in X$.

**Lemma 2.1** [11] (Mean-value theorem) Let $x^1, x^2 \in X$ and $f : X \to R$ be locally Lipschitz continuous near each point of a nonempty closed convex set containing the line segment $[x^1, x^2]$. Then there exist a point $u \in (x^1, x^2)$ and $\xi \in \partial^c f(u)$ such that $f(x^1) - f(x^2) = \langle \xi, x^1 - x^2 \rangle$.

In following, we will introduce the concepts of pseudo (quasi) B-invexity and of invariant pseudo (quasi) B-monotonicity.

**Definition 2.1** Let $b : K \times K \times [0,1] \to R$, be a function. The function $f$ is said to be

(i)[5,6] B-preinvex on $K$ with respect to $\eta$ and $b$ if for any $x^1, x^2 \in K$ and any $\lambda \in [0,1]$ one has $f(x^2 + \lambda \eta(x^1, x^2)) \leq b(x^1, x^2, \lambda) f(x^1) + (1 - b(x^1, x^2, \lambda)) f(x^2)$;

(ii) quasi B-preinvex on $K$ with respect to $\eta$ and $b$ if for any $x^1, x^2 \in K$ and any $\lambda \in [0,1]$ one has $f(x^1) \leq f(x^2)$ implies $b(x^1, x^2, \lambda) f(x^2 + \lambda \eta(x^1, x^2)) \leq b(x^1, x^2, \lambda) f(x^2)$. 

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From Definition 2.1, we can easily see that B-preinvexity implies quasi B-preinvexity with respect to the same \( \eta \) and \( b \). But the converse is not necessarily true, see the following example.

**Example 2.1** Let \( X = \mathbb{R} \) and \( K = \left[ 0, \frac{\pi}{2} \right] \). For any \( x \in X \), \( x', x^2 \in K \), and \( \lambda \in [0, 1] \), let \( \eta(x', x^2) = \sin x' - \sin x^2 \) and

\[
f(x) = \begin{cases} 
    x, & 0 < x < \frac{\pi}{2}, \\
    1, & x = \frac{\pi}{2}, \\
    0, & \text{otherwise},
\end{cases}
\]

\[
b(x', x^2, \lambda) = \begin{cases} 
    0, & x' = \frac{\pi}{2} \text{ or } x^2 = \frac{\pi}{2} \text{ or } \lambda = 0, \\
    1, & \text{otherwise}.
\end{cases}
\]

We can verify that \( f \) is quasi B-preinvex on \( K \) with respect to \( \eta \) and \( b \). However, for \( x' = \frac{\pi}{2} \) and \( x^2 = \frac{\pi}{2} \), we have \( f(x' + \lambda \eta(x', x^2)) = \frac{\pi}{2} - \frac{\sqrt{2}}{4} \) and \( \lambda b(x', x^2, \lambda) f(x') + (1 - \lambda b(x', x^2, \lambda)) f(x^2) = 1 \), which indicates that \( f \) is not B-preinvex on \( K \) with respect to \( \eta \) and \( b \).

**Definition 2.2** Let \( b : K \times K \to \mathbb{R} \) be a function. \( f \) is said to be

(i) B-invex on \( K \) with respect to \( \eta \) and \( b \) if for any \( x', x^2 \in K \) and any \( \xi \in \partial f(x^2) \) one has

\[
\langle \xi, \eta(x', x^2) \rangle \leq b(x', x^2) (f(x') - f(x^2));
\]

(ii) quasi B-invex on \( K \) with respect to \( \eta \) and \( b \) if for any \( x', x^2 \in K \) and any \( \xi \in \partial f(x^2) \) one has

\[
f'(x') \leq f(x^2) \implies b(x', x^2) \langle \xi, \eta(x', x^2) \rangle \leq 0;
\]

(iii) pseudo B-invex on \( K \) with respect to \( \eta \) and \( b \) if for any \( x', x^2 \in K \) and some \( \xi \in \partial f(x^2) \) one has

\[
\langle \xi, \eta(x', x^2) \rangle \geq 0 \implies b(x', x^2) f'(x') \geq b(x', x^2) f(x^2).
\]

From Definition 2.2, we can see that B-invexity implies quasi or pseudo B-invexity with respect to the same \( \eta \) and \( b \). But the converses are not necessarily true, see the following example.

**Example 2.2** Let \( X = \mathbb{R} \) and \( K = \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \). For any \( x \in X \) and \( x', x^2 \in K \) let \( f(x) = |x|, \eta(x', x^2) = \sin x' - \sin x^2 \) and

\[
b(x', x^2) = \begin{cases} 
    1, & x'x^2 > 0, \\
    0, & x'x^2 \leq 0.
\end{cases}
\]

We can verify that \( f \) is quasi and pseudo B-invex on \( K \) with respect to \( \eta \) and \( b \). However, for \( x' = -\frac{\pi}{4}, x^2 = 0 \) and \( \xi = -1 \in \partial f(x') \) it follows that \( \langle \xi, \eta(x', x^2) \rangle > b(x', x^2) (f(x') - f(x^2)) \), which indicates that \( f \) is not B-invex on \( K \) with respect to \( \eta \) and \( b \).

**Definition 2.3** Let \( b : K \times K \to \mathbb{R} \) be a function and \( F : X \to 2^X \) be a set-valued mapping. \( F \) is said to be

(i) invariant B-monotone on \( K \) with respect to \( \eta \) and \( b \) if for any \( x', x^2 \in K \) and any \( u \in F(x^2), v \in F(x') \) one has

\[
b(x', x^2) \langle u, \eta(x', x^2) \rangle + b(x', x^2) \langle v, \eta(x', x^2) \rangle \leq 0;
\]
(ii) invariant quasi B-monotone on $K$ with respect to $\eta$ and $b$ if for any $x^1, x^2 \in K$, some $v \in F(x^1)$ and any $u \in F(x^2)$ one has $b(x^2, x^2)^T \langle v, \eta(x^1, x^2) \rangle > 0$ implies $b(x^1, x^1)^T \langle v, \eta(x^1, x^1) \rangle \leq 0$;

(iii) invariant pseudo B-monotone on $K$ with respect to $\eta$ and $b$ if for any $x^1, x^2 \in K$, some $v \in F(x^1)$ and any $u \in F(x^2)$ one has $b(x^2, x^2)^T \langle v, \eta(x^1, x^2) \rangle > 0$ implies $b(x^1, x^1)^T \langle v, \eta(x^1, x^1) \rangle \leq 0$.

From Definition 2.3, we can see that invariant B-monotonicity implies invariant quasi B-monotonicity and invariant pseudo B-monotonicity implies invariant quasi B-monotonicity. But the converses are not necessarily true, see the following two examples.

Example 2.3 Let $X = R$ and $K = \left( \frac{\pi}{2} \right)$. For any $x \in X$ and $x^1, x^2 \in K$ let $f(x) = |x|, \eta(x^1, x^2) = \sin x^1 - \sin x^2$ and

$$b(x^1, x^2) = \begin{cases} 1, x^2 > 0, \\ 0, x^2 \leq 0. \end{cases}$$

We can verify that $\partial^c f$ is invariant quasi B-monotone on $K$ with respect to $\eta$ and $b$. However, for $x^1 = 0, x^2 = \frac{\pi}{4}, u = 1 \in \partial^c f(x^1)$ and any $v \in \partial^c f(x^2)$, due to $b(x^2, x^2)^T \langle v, \eta(x^1, x^2) \rangle + b(x^1, x^1)^T \langle v, \eta(x^1, x^1) \rangle > 0$, we can conclude that $\partial^c f$ is not invariant B-monotone on $K$ with respect to $\eta$ and $b$.

Example 2.4 Let $X = R$ and $K = \left( \frac{\pi}{2} \right)$. For any $x \in X$ and $x^1, x^2 \in K$ let $f(x) = -|x|, \eta(x^1, x^2) = \sin x^1 - \sin x^2$ and

$$b(x^1, x^2) = \begin{cases} 1, x^1 > 0, x^2 = 0, \\ 0, \text{ otherwise.} \end{cases}$$

We can verify that $\partial^c f$ is invariant quasi B-monotone on $K$ with respect to $\eta$ and $b$. However, for $x^1 = \frac{\pi}{4}, x^2 = 0, u = -1 \in \partial^c f(x^1)$ and any $v \in \partial^c f(x^2)$, we can conclude that $b(x^2, x^2)^T \langle v, \eta(x^1, x^2) \rangle \geq 0$ implies $b(x^1, x^1)^T \langle v, \eta(x^1, x^1) \rangle > 0$. Hence, $\partial^c f$ is not invariant pseudo B-monotone on $K$ with respect to $\eta$ and $b$.

3 Relations between (Quasi) B-preinvexity and (Pseudo, Quasi) B-invexity

In this section, we mainly study the relations between (quasi) B-preinvexity and (pseudo, quasi) B-invexity for a locally Lipschitz continuous function $f : X \rightarrow R$. For this purpose, we need the following assumptions, which are taken from [9].

Assumption A $f(x^2 + \eta(x^1, x^2)) \leq f(x^1), \forall x^1, x^2 \in K$.

Assumption C For any $x^1, x^2 \in K$ and any $\lambda \in [0, 1]$, one has $\eta(x^2, x^2 + \lambda \eta(x^1, x^1)) = -\lambda \eta(x^1, x^1)$ and $\eta(x^1, x^1 + \lambda \eta(x^1, x^1)) = (1 - \lambda) \eta(x^1, x^1)$.

Yang et al. [9] showed that if $\eta$ satisfies Assumption C, then $\eta(x^2 + \lambda \eta(x^1, x^1), x^2) = \lambda \eta(x^1, x^1)$ for all $x^1, x^2 \in K$ and $\lambda \in [0, 1]$.

Theorem 3.1 Let

(i) $b : K \times K \times [0, 1] \rightarrow R$ be such that $b(x^1, x^2, \cdot)$ is continuous at $0^+$ for any fixed $x^1, x^2 \in K$;
(ii) $\eta$ and $b$ be continuous with respect to the second argument, respectively;
(iii) $\overline{b}$ be bounded, where $\overline{b}(x',x^2) = \lim_{\delta \downarrow 0} b(x',x^2,\lambda)$ for all $x',x^2 \in K$.

If $f$ is B-preinvex on $K$ with respect to $\eta$ and $b$, then $f$ is B-invex on $K$ with respect to $\eta$ and $\overline{b}$.

But the converse is not necessarily true.

**Proof:** For any given $x', x^2 \in K$ and $\epsilon > 0$, let $L$ be the local Lipschitz constant of $f$ at $x^2$. Then there exists a constant $0 < \delta < \frac{\epsilon}{2L}$ such that $|f(x') - f(x)| < \frac{\epsilon}{2}$ and $\|f(x',x^2) - f(x')\| < \frac{\epsilon}{2L}$ for all $x \in K$ with $|x^2 - x| < \delta$.

Consequently, for a small enough number $\lambda > 0$, one has

$$
\frac{f(x + \lambda \eta(x',x^2)) - f(x)}{\lambda} \\
\leq \frac{f(x + \lambda \eta(x',x)) - f(x)}{\lambda} + L\|\eta(x',x^2) - \eta(x',x)\| \\
\leq \frac{\lambda b(x',x,\lambda) f(x') + (1 - \lambda b(x',x,\lambda)) f(x) - f(x)}{\lambda} + L\|\eta(x',x^2) - \eta(x',x)\| \\
\leq b(x',x,\lambda)(f(x') - f(x)) + \frac{\epsilon}{2}(b(x',x,\lambda) + 1).
$$

Taking the limit as $\lambda \downarrow 0, \epsilon \downarrow 0$ and $x \to x^2$, since $\overline{b}$ is bounded, we get $\langle \xi, \eta(x',x^2) \rangle \leq f''(x^2; \eta(x',x^2)) \leq \overline{b}(x',x^2)(f(x') - f(x^2))$ for all $\xi \in \partial f'(x^2)$, which shows that $f$ is B-invex on $K$ with respect to $\eta$ and $\overline{b}$.

The following example shows that the converse is not true.

**Example 3.1** Let $X = R$ and $K = \left\{ -\frac{\pi}{2}, \frac{\pi}{2} \right\}$. For any $x \in X, x', x^2 \in K$ and $\lambda \in [0,1]$, let $\eta(x',x^2) = \frac{x^2 - x^2}{3}$ and

$$f(x) = \begin{cases} 
3x, & x \geq 0, \\
0, & x < 0,
\end{cases}
$$

$$b(x',x^2,\lambda) = \begin{cases} 
1, & x^2 = 0, x' > 0, \\
\lambda, & x^2 = 0, x' \leq 0, \\
\frac{1}{3}x^2, & x' \neq 0.
\end{cases}
$$

Then

$$\overline{b}(x',x^2) = \begin{cases} 
1, & x^2 = 0, x' > 0, \\
0, & x^2 = 0, x' \leq 0, \\
\frac{1}{3}x^2, & x' \neq 0.
\end{cases}
$$

We can verify that $f$ is B-invex on $K$ with respect to $\eta$ and $\overline{b}$. However, for $x' = -\frac{\pi}{4}, x^2 = 0$ and $\lambda = \frac{1}{2}$, we can deduce that $f(x^2 + \lambda \eta(x',x^2)) = \lambda b(x',x^2,\lambda) f(x') + (1 - \lambda b(x',x^2,\lambda)) f(x^2)$. Hence,
f is not B-preinvex on K with respect to η and b.

**Theorem 3.2** Let \( b : K \times K \to R_+ \). If \( f \) is B-invex on K with respect to \( η \) and \( b \) and satisfies Assumption C, then \( f \) is B-preinvex on K with respect to \( η \) and \( b \), where

\[
\overline{b}(x^1, x^2, λ) = \frac{b(x^1, x^2 + λη(x^1, x^2))}{λb(x^1, x^2 + λη(x^1, x^2)) + (1 - λ)b(x^1, x^2 + λη(x^1, x^2))}
\]

for all \( x^1, x^2 \in K \) and \( λ \in [0,1] \).

**Proof:** Take arbitrarily \( x^1, x^2 \in K \) and \( λ \in [0,1] \) and let \( x^0 = x^2 + λη(x^1, x^2) \). By the definition of B-invexity, for any \( ξ \in \partial f(x^0) \), we have

\[
\langle ξ, η(x^1, x^2) \rangle \leq b(x^1, x^0)(f(x^1) - f(x^0)), \tag{1}
\]

\[
\langle ξ, η(x^2, x^0) \rangle \leq b(x^2, x^0)(f(x^2) - f(x^0)). \tag{2}
\]

Multiplying (1) by \( λ \) and (2) by \( (1 - λ) \) and adding them, by Assumption C, we can deduce that \( λb(x^1, x^0)f(x^1) + (1 - λ)b(x^2, x^0)f(x^2) \geq (λb(x^1, x^0) + (1 - λ)b(x^2, x^0))f(x^0) \), which implies that \( f(x^1 + λη(x^1, x^2)) \leq λ\overline{b}(x^1, x^2, λ)f(x^1) + (1 - λ\overline{b}(x^1, x^2, λ))f(x^2) \), where

\[
\overline{b}(x^1, x^2, λ) = \frac{b(x^1, x^2 + λη(x^1, x^2))}{λb(x^1, x^2 + λη(x^1, x^2)) + (1 - λ)b(x^1, x^2 + λη(x^1, x^2))}.
\]

Therefore, the assertion of the theorem holds.

The following two examples show that there are not direct implications between quasi B-invexity and quasi B-preinvexity.

**Example 3.2** Let \( X = R \) and \( K = \left(-\frac{π}{2}, \frac{π}{2}\right) \). For any \( x \in X, x^1, x^2 \in K \) and \( λ \in [0,1] \) let \( f(x) = -|x| \) and

\[
η(x^1, x^2) = \begin{cases} 
\sin x^1 - \sin x^2, & x^1x^2 \geq 0, \\
0, & x^1x^2 < 0,
\end{cases}
\]

\[
b(x^1, x^2, λ) = \begin{cases} 
1, & x^1x^2 \geq 0, \\
λ, & x^1x^2 < 0.
\end{cases}
\]

Then

\[
\overline{b}(x^1, x^2) = \lim_{h \to 0} b(x^1, x^2, λ) = \begin{cases} 
1, & x^1x^2 \geq 0, \\
0, & x^1x^2 < 0.
\end{cases}
\]

We can verify that \( f \) is quasi B-preinvex on K with respect to \( η \) and \( b \). For \( x^1 = -\frac{π}{4}, x^2 = 0 \) and \( ξ = -1 \in \partial f(x^1) \), we can deduce that \( f(x^1) \leq f(x^2) \) implies \( \overline{b}(x^1, x^2)\langle ξ, η(x^1, x^2) \rangle > 0 \). Hence, \( f \) is not quasi B-invex on K with respect to \( η \) and \( b \).
Example 3.3 $X = R$ and $K = \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$. For any $x \in X, x' \in K$ and $\lambda \in [0,1]$ let $f(x) = -|x|$, $\eta(x', x^2) = \sin x^1 - \sin x^2$ and

$$
\begin{aligned}
\tau(x', x^2, \lambda) &= \begin{cases}
1, & x^1 x^2 > 0, \\
0, & x^1 x^2 \leq 0.
\end{cases}
\end{aligned}
$$

Then

$$
\tau(x', x^2) = \lim_{\lambda \downarrow 0} \tau(x', x^2, \lambda) = \begin{cases}
1, & x^1 x^2 > 0, \\
0, & x^1 x^2 \leq 0.
\end{cases}
$$

We can verify that $f$ is quasi B-invex on $K$ with respect to $\eta$ and $\tau$. However, for $x^1 = \frac{\pi}{4}, x^2 = -\frac{\pi}{6}$ and $\lambda = \frac{1}{2}$, since $f(x^1) \leq f(x^2)$ implies $\tau(x^1, x^2, \lambda) f(x^1 + \lambda \eta(x^1, x^2)) > \tau(x^1, x^2, \lambda) f(x^2)$, we can conclude that $f$ is not quasi B-preinvex on $K$ with respect to $\eta$ and $\tau$.

**Theorem 3.3** Let

(i) $b : K \times K \times [0,1] \rightarrow R_+$ be such that $b(x', x^2, \cdot)$ is continuous at $0^+$ for any fixed $x^1, x^2 \in K$;

(ii) $\eta$ and $b$ be continuous with respect to the second argument, respectively;

(iii) $b$ be bounded, where $b(x', x^2) = \lim_{x \rightarrow x', \lambda \downarrow 0} b(x^2, x, \lambda)$ for all $x^1, x^2 \in K$.

If $f$ is quasi B-preinvex on $K$ with respect to $\eta$ and $b$, then for any $\xi \in \partial^* f(x^1)$ we have

$$
f(x^2) < f(x^1) \Rightarrow \tau(x^1, x^2) \left\{ \xi, \eta(x^2, x^1) \right\} \leq 0.
$$

**Proof:** Let $x^1, x^2 \in K$ with $f(x^2) < f(x^1)$ and $L > 0$ be the local Lipschitz constant of $f$ at $x^1$. By the continuity of $f$ at $x^1$, we know that there exists a constant $\delta > 0$ such that $f(x^2) < f(x)$ for all $x \in X$ with $\|x - x^1\| < \delta$. For any $\lambda \in [0,1]$ and any $x \in K$ with $\|x - x^1\| < \delta$, by the quasi B-preinvexity of $f$, we have $b(x^2, x, \lambda)f(x + \lambda \eta(x^2, x)) \leq b(x^2, x, \lambda)f(x)$. Consequently,

$$
\begin{aligned}
b(x^2, x, \lambda)f(x + \lambda \eta(x^2, x)) - f(x) \\
&\leq b(x^2, x, \lambda)\left\{ \frac{f(x + \lambda \eta(x^2, x)) - f(x)}{\lambda} + L\|\eta(x^2, x) - \eta(x^1, x^1)\| \right\} \\
&\leq b(x^2, x, \lambda)L\|\eta(x^2, x) - \eta(x^1, x^1)\|.
\end{aligned}
$$

Taking the limit for the last inequality as $x \rightarrow x^1$ and $\lambda \downarrow 0$, since $b$ is bounded, we get $\tau(x^2, x') f^0(x^1; \eta(x^2, x')) \leq 0$ and then $\tau(x^2, x') \left\{ \xi, \eta(x^2, x^1) \right\} \leq 0, \forall \xi \in \partial^* f(x^1)$, which indicates that the assertion of the theorem holds.

**Theorem 3.4** Let $b : K \times K \rightarrow R_+$ be continuous with respect to the second argument. If $f$ is quasi B-invex on $K$ with respect to $\eta$ and $b$ and satisfies Assumption C, then $f$ is quasi B-preinvex on $K$ with respect to $\eta$ and $b$, where $b(x^2, x^2, \lambda) = b(x^1, x^2 + \lambda \eta(x^1, x^2))b(x^2, x^2 + \lambda \eta(x^1, x^2))$ for all $x^1, x^2 \in K$ and $\lambda \in [0,1]$.

**Proof:** Take arbitrarily $x^1, x^2 \in K$ and let $f(x^1) \leq f(x^2)$. In order to show that $f$ is quasi B-preinvex on $K$, it suffices to prove that the set
\[ \Omega = \{ x^2 + \lambda \eta(x^1, x^2) : \bar{f}(x^1, x^2, \lambda) f(x^1, x^2, \lambda) > \bar{f}(x^1, x^2, \lambda) f(x^1, x^2, \lambda), \lambda \in [0, 1] \} \]

is empty. It is evident that \( \Omega \) is equivalent to the set
\[ \Omega' = \{ x^2 + \lambda \eta(x^1, x^2) : f(x^2 + \lambda \eta(x^1, x^2)) > f(x^2), \bar{f}(x^1, x^2, \lambda) > 0, \lambda \in [0, 1] \} . \]

Assume to the contrary that \( \Omega' \neq 0 \). By the continuity of \( f \), we know that the set
\[ \Omega'' = \{ x^2 + \lambda \eta(x^1, x^2) : f(x^2 + \lambda \eta(x^1, x^2)) > f(x^2), \bar{f}(x^1, x^2, \lambda) > 0, \lambda \in [0, 1] \} \]
is nonempty. Consequently, for every \( \tau \in \Omega'' \), there exist \( \lambda \in (0, 1) \) such that \( \tau = x^2 + \lambda \eta(x^1, x^2) \), \( \bar{f}(x^1, x^2, \lambda) > 0 \) and \( f(\bar{\tau}) > f(\tau) \). For any \( \xi \in \partial^c f(\bar{\tau}) \), by the quasi-B-invexity of \( f \), it follows that
\[ \begin{align*}
\langle \xi, \eta(x^1, x^2) \rangle &\leq 0 \quad \text{and} \quad b(x^1, x^2) \langle \xi, \eta(x^1, x^2) \rangle = 0,
\end{align*} \]
which together with Assumption C shows that \( b(x^1, x^2)\eta(x^1, x^2) = 0 \), which is a contradiction. Hence, the assertion of the theorem holds.

### 4 Relations between Generalized B-invexity and Generalized Invariant B-monotonicity

In this section, we mainly study the relations between (pseudo, quasi) B-invexity of a locally Lipschitz continuous function \( f \) and invariant (pseudo, quasi) B-monotonicity of its subdifferential mapping \( \partial^c f \).

The following result is a direct consequence of Definition 2.1.

**Theorem 4.1** Let \( b : K \times K \to R \). If \( f \) is B-invex on \( K \) with respect to \( \eta \) and \( b \), then \( \partial^c f \) is invariant B-monotone on \( K \) with respect to the same \( \eta \) and \( b \).

**Theorem 4.2** Let \( b : K \times K \to R_+ \). If \( \partial^c f \) is invariant B-monotone on \( K \) with respect to \( \eta \) and \( b \) and satisfies Assumptions A and C, then there exists a function \( \lambda : K \times K \to (0, 1) \) such that \( f \) is B-invex on \( K \) with respect to \( \eta \) and \( \bar{b} \), \( \bar{b}(x^1, x^2) = \frac{b(x^1, x^2 + \lambda(x^1, x^2) \eta(x^1, x^2), x^1)}{b(x^1, x^2 + \lambda(x^1, x^2) \eta(x^1, x^2), x^1)} \) for all \( x^1, x^2 \in K \).

**Proof:** Let \( \partial^c f \) be invariant B-monotone on \( K \). For any \( x^1, x^2 \in K \), by Assumption A and Lemma 2.1, there exist a constant related to \( x^1, x^2 \) in \( (0, 1) \), denoted by \( \lambda(x^1, x^2) \), and a point \( \bar{\xi} \in \partial^c f(x^0) \) such that
\[ f(x^1) - f(x^2) \leq f(x^1) - f(x^2 + \eta(x^1, x^2)) = -\langle \bar{\xi}, \eta(x^2, x^1) \rangle, \]
where \( x^0 = x^1 + \lambda(x^1, x^2) \), \( \eta(x^1, x^2) \). By Assumption C, for any \( v \in \partial^c f(x^1) \), it follows that \( b(x^0, x^1)\langle \bar{\xi}, \eta(x^1, x^2) \rangle \geq b(x^1, x^0)\langle u, \eta(x^1, x^2) \rangle \)
and then
\[ b(x^0, x^1)(f(x^2) - f(x^1)) \geq b(x^1, x^0)\langle u, \eta(x^1, x^2) \rangle \]
which is equivalent to 

\[ \frac{b(x^0, x^1)}{b(x^0, x^0)} (f(x^0) - f(x^1)) \geq \langle \nu, \eta(x^0, x^1) \rangle. \]

The last inequality shows that \( f \) is B-invex on \( K \) with respect to \( \eta \) and \( \bar{b} \).

The following result is a direct consequence of Definition 2.2.

**Theorem 4.3** Let \( b : K \times K \rightarrow \mathbb{R}_+ \). If \( f \) is quasi B-invex on \( K \) with respect to \( \eta \) and \( b \), then \( \partial^c f \) is invariant quasi B-monotone on \( K \) with respect to \( \eta \) and \( \bar{b} \), where \( \bar{b}(x^1, x^2) = b(x^2, x^1) \) for all \( x^1, x^2 \in K \). But the converse is not necessarily true for same \( \eta \) and \( b \).

**Example 4.1** Let \( X, K, f, \eta \) and \( b \) be the same as in Example 2.4. Then \( \partial^c f \) is invariant quasi B-monotone on \( K \) with respect to \( \eta \) and \( b \). However, for \( x^1 = -\pi, x^2 = 0 \) and \( \xi = -1 \in \partial^c f(x^2) \), we can deduce that \( f(x^1) \leq f(x^2) \) implies \( b(x^1, x^2)((\xi, \eta(x^1, x^2))) = 4 \). This shows that \( f \) is not quasi B-invex on \( K \) with respect to \( \eta \) and \( b \).

**Theorem 4.4** Let \( b : K \times K \rightarrow \mathbb{R} \). If \( \partial^c f \) is invariant pseudo B-monotone on \( K \) with respect to \( \eta \) and \( b \) and satisfies Assumptions A and C, then there exists a function \( \lambda : K \times K \rightarrow (0, 1) \) such that \( f \) is pseudo B-invex on \( K \) with respect to \( \eta \) and \( \bar{b} \), where \( \bar{b}(x^1, x^2) = b(x^2 + \lambda(x^1, x^2)) \eta(x^1, x^2) \) for all \( x^1, x^2 \in K \).

**Proof:** Take arbitrarily \( x^1, x^2 \in K \). For \( x^2 + \eta(x^1, x^2) \), by Lemma 2.1, there exist a constant related to \( x^1, x^2 \) in \( (0, 1) \), denoted by \( \lambda(x^1, x^2) \), and a point \( u \in \partial^c f(x^2 + \lambda(x^1, x^2)) \eta(x^1, x^2) \) such that

\[ f(x^2 + \eta(x^1, x^2)) - f(x^2) = \langle u, \eta(x^1, x^2) \rangle, \quad (4) \]

Assume to the contrary that the assertion of the theorem is not true. Then there exist \( x^1, x^2 \in K \) such that

\[ \langle u, \eta(x^1, x^2) \rangle \geq 0, \forall \nu \in \partial^c f(x^2), \quad (5) \]

and \( \bar{b}(x^1, x^2) f(x^1) < \bar{b}(x^1, x^2) f(x^2) \), which shows that \( f(x^1) < f(x^2) \). By Assumption A and (4), we have \( b(x^2 + \lambda(x^1, x^2)) \eta(x^1, x^2) u(x^2 + \lambda(x^1, x^2)) \eta(x^1, x^2) < 0 \). By Assumption C, we get \( b(x^2 + \lambda(x^1, x^2)) \eta(x^1, x^2) \langle u, \eta(x^2 + \lambda(x^1, x^2)) \eta(x^1, x^2) \rangle > 0 \) By the invariant pseudo B-monotonicity of \( \partial^c f \), for some \( \omega \in \partial^c f(x^2) \), we obtain

\[ b(x^2 + \lambda(x^1, x^2)) \eta(x^1, x^2) \omega(x^1, x^2) \langle \omega, \eta(x^1, x^2) \rangle < 0 \]

which implies that \( \langle \omega, \eta(x^1, x^2) \rangle < 0 \). This contradicts (5). Therefore, the assertion of the theorem holds.

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References