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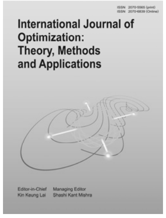

Some Applications of Invexity and Generalized Invexity to Pareto Optimization Problems

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Abstract. We consider three applications of invexity and generalized invexity to a multiobjective (Pareto) optimization problem: i) It is shown that in a Pareto optimization problem involving generalized pseudolinear functions, every efficient solution is a properly efficient solution under some boundedness condition. ii) Some remarks are made on semilocally preinvex functions, their generalizations and on applications. iii) We recall the definition of a class of generalized invex functions by means of *K*-dimensional derivatives; then we apply this class of functions to a Pareto optimization problem.

Keywords: multiobjective optimization problems, invexity.

1 Introduction

We consider the following three applications of invexity and generalized invexity to a multiobjective (Pareto) optimization problem.

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1. It is shown that in a Pareto optimization problem involving generalized pseudolinear functions (i.e. η-pseudolinear functions), every efficient solution is properly efficient under some boundedness condition. This approach generalizes similar results of Chew and Choo [6].
2. We make some remarks on semilocally convex, semilocally preinvex and semilocally invex functions, together with an application in obtaining sufficient optimality conditions for a Pareto optimization problem.
3. Following the approach of Castellani [5], we recall the definition of a class of generalized invex functions by means of K-dimensional derivatives, when K is a local cone approximation. Then we apply this class of functions in obtaining optimality results for a Pareto optimization problem.

Hanson [22] presented a weakened concept of convexity for differentiable functions, i.e. the class of invex functions.

Definition 1. *A differentiable function $f: X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is an open set, is said to be invex if there exists a function $\eta: X \times X \to \mathbb{R}^n$ such that*

$$f(x) - f(y) \geq \eta(x, y)\nabla f(y), \forall x, y \in X.$$

The name "invex" stems from a contraction of "invariant convex" and was proposed by Craven [8].

Since invexity requires differentiability, Ben-Israel and Mond [3] and Weir and Mond [39] introduced the following class of functions. Let f be a real-valued function defined on a subset of $\mathbb{R}^n$ and let $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. We say that a subset X of $\mathbb{R}^n$ is η-invex if for every $x, y \in X$ the segment $[y, y + \eta(x, y)]$ is contained in X.

Definition 2. *Let $f: X \to \mathbb{R}$ be defined on the η-invex set $X \subseteq \mathbb{R}^n$. We say that f is preinvex with respect to η if*

$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1-\lambda) f(y), \forall x, y \in X, \forall \lambda \in [0,1]. \tag{1}$$

A differentiable function satisfying (1) is also invex. For some recent considerations on preinvex functions the reader may consult Giorgi ([15],[16]). Basic papers are Hanson [22], Kaul and Kaur [25], Ben-Israel and Mond [3] and Pini [31].

Invex functions, their generalizations and restrictions have been extensively applied to a wide class of optimization problems, both static and dynamic, both in the scalar case and in the vectorial case. See, e.g., Giorgi and Mishra [27] for a recent survey.

Consider now the following basic Pareto multiobjective optimization problem.

$$\text{(VOP)} \;\; \text{V-minimize} \;\; f(x) = (f_1(x), \cdots, f_p(x))$$

$$\text{subject to } x \in X, g_j(x) \leq 0, j = 1, \cdots, m,$$

where $X \subseteq \mathbb{R}^n$ is a nonempty open set, and $f_i, g_j: \mathbb{R}^n \to \mathbb{R}, \forall i \in I = \{1, \cdots, p\}, \forall j \in J = \{1, \cdots, m\}$. $J(x_0) = \{j \in J : g_j(x_0) = 0\}$ is the index set of the active constraints at x_0 and $S = \{x \in X : g_j(x) \leq 0, j \in J\}$ is the feasible set of (VOP).

We accept the following usual definitions.

Definition 3. *For problem (VOP), a point* $x_0 \in S$ *is said to be a weak minimum point (or a weak efficient point) if there exists no other feasible point* $x \in S$ *such that* $f_i(x) < f_i(x_0), \forall i \in I$. *A point* $x_0 \in S$ *is said to be a minimum point (or efficient point) if there exists no other feasible point* $x \in S$ *such that for some* $s = 1, \cdots, p$ *we have* $f_s(x) < f_s(x_0)$ *and* $f_i(x) \leq f_i(x_0), \forall i \neq s$ *(in other words,* $f(x) \leq f(x_0)$ *implies that* $f(x_0) = f(x)$*).*

A basic result of Ben-Israel and Mond [3] and Craven and Glover [10] is that a scalar function is invex if and only if every stationary point of *f* is a global minimum point for *f*. This property is lost for a vector invex function (i.e. a function whose components are invex with respect to the same η), as shown by Cambini and Martein [4].

Here we accept the following definition of stationary point for a vector valued function: the point $x_0 \in S$ is a stationary point for *f* if there exists a vector $p \in \mathbb{R}_+^p \setminus \{0\}$, such that $p \nabla f(x_0) = 0$.

We note, however, that by means of a particular definition of invexity, introduced by Jeyakumar and Mond [23], it is possible to get for (VOP) something similar to the scalar case.

Definition 4. *Let* $X \subseteq R^n$. *A vector-valued function* $f : X \to \mathbb{R}^p$ *is said to be V-invex if there exist functions* $\eta : X \times X \to \mathbb{R}^n$ *and* $\alpha_i : X \times X \to \mathbb{R}_+ \setminus \{0\}$ *such that, for each* $x, y \in X$ *and for* $i = 1, \cdots, p$

$$f_i(x) - f_i(y) \geq \alpha_i(x, y)\eta(x, y)\nabla f_i(y). \tag{2}$$

Jeyakumar and Mond [23] prove the following result.

Theorem 1. *Let* $f : \mathbb{R}^n \to \mathbb{R}^p$ *be V-invex. Then* x_0 *is a (global) weak minimum point for f if and only if it is a stationary point for f, i.e. there exists* $p \in \mathbb{R}_+^p \setminus \{0\}$*, such that* $\sum_{i=1}^{p} p_i \nabla f_i(x_0) = 0$.

Hanson [22] and Kaul and Kaur [25] introduced also the definitions of pseudoinvex functions and quasiinvex functions. It is seen (see Ben-Israel and Mond [3]) that, for the scalar case, the class of invex functions coincides with the class of pseudoinvex functions. Taking into consideration the following "vector definition" of pseudoinvexity, this is no longer true.

Definition 5. *Let* $f : X \to \mathbb{R}^p$ *be a differentiable function on the open set* $X \subseteq \mathbb{R}^n$. *Then f is vector-pseudoinvex on X if there exists a function* $\eta : X \times X \to \mathbb{R}^n$ *such that*

$$f(x) - f(y) < 0 \Rightarrow \nabla f(y)\eta(x, y) < 0, \forall x, y \in X.$$

We note that the above definition is not equivalent to require the pseudoinvexity, with respect to η, of each component of *f*. By means of Definition 5, Osuna-Gómez et al. [30] obtain the following interesting result.

Theorem 2. *A vector-valued differentiable function* $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ *is vector-pseudoinvex on X if and only if every (vector) stationary point of f is a weak efficient point for f.*

2 η-Pseudolinearity and Efficiency

We have already remarked, in the previous section, that Hanson [22] also introduced other more general classes of functions, with respect to the invex functions. In particular, he defined as follows the class of pseudoinvex (or η-pseudoconvex) functions:

$$\eta(y,x)\nabla f(x) \geq 0 \Rightarrow f(y) \geq f(x), \forall x, y \in X,$$

where $f: X \to \mathbb{R}$ and $X \subseteq \mathbb{R}^n$.

We have also noted that, unlike convex and pseudoconvex functions, the class of (scalar) invex functions and the class of pseudoinvex functions coincide.

Definition 6. *(Ansari et al. [1]) A differentiable function defined on an open set $X \subseteq \mathbb{R}^n$ is called η-pseudolinear if f and $-f$ are η-pseudoconvex with respect to the same η.*

If $\eta(y,x) = y - x$, we have the pseudolinear functions, studied by Chew and Choo [6] who found conditions for an efficient solution of a nonlinear vector optimization problem to be proper efficient.

Definition 7. *(Mohan and Neogy [28]). The vector-valued function $\eta: X \times X \to \mathbb{R}^n$, $X \subseteq \mathbb{R}^n$, satisfies condition C if for any $x, y \in X$*

$$\eta(x, x+\lambda\eta(y,x)) = -\lambda\eta(y,x), \eta(y, x+\lambda\eta(y,x)) = (1-\lambda)\eta(y,x), \forall\lambda \in [0,1].$$

The following result is due to Ansari et al. [1].

Theorem 3. *(i) Suppose that $f: X \to \mathbb{R}$ is η-pseudolinear satisfying condition C. Then for all $x, y \in X$ it holds $\eta(x,y)\nabla f(x) = 0$ if and only if $f(y) = f(x)$.*

(ii) A differentiable function $f: X \to \mathbb{R}$ is η-pseudolinear if and only if there exists a real-valued function p defined on $X \times X$ such that $p(x,y) > 0$ and

$$f(y) = f(x) + p(x,y)\eta(x,y)\nabla f(x), \forall x, y \in X. \quad (3)$$

Let now consider problem (VOP) and assume that the differentiable functions $f_i, i \in I$, and g_j, $j \in J$, are η-pseudolinear on the open set $X \subseteq \mathbb{R}^n$, with proportional functionals p_i and q_j, respectively (see part (ii) of Theorem 3). We need the following definition of properly efficient solution for (VOP), due to Geoffrion [14].

Definition 8. *A feasible point x is properly efficient for (VOP) if it is efficient and there exists a real number $M > 0$ such that, for each $i \in I$ we have that*

$$f_i(y) - f_i(x) \geq M(f_j(x) - f_j(y))$$

for some j such that $f_j(x) < f_j(y)$ whenever $f_i(y) < f_i(x)$.

The following results are proved by Giorgi and Rueda [21] and generalize to η-pseudolinearity some propositions of Chew and Choo [6].

Theorem 4. *Consider problem (VOP), where the differentiable functions $f_i, i \in I$ and $g_j, j \in J$, are η-pseudolinear on the set $X \subseteq \mathbb{R}^n$ with proportional functionals p_i and q_j, respectively. Let* *condition C be satisfied for all* $x, y \in X$. *A feasible point* x_0 *is and efficient solution of (VOP) if and only if there exist multipliers* $\lambda_i > 0, i \in I$, *and* $\mu_j \ge 0, j \in J(x_0)$, *such that*

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x_0) + \sum_{j \in J(x_0)} \mu_j \nabla g_j(x_0) = 0. \tag{4}$$

Definition 9. *(Chew and Choo [6]). A feasible point* x_0 *is said to satisfy the boundedness condition if the set*

$$\left\{ \frac{p_i(x_0, x)}{p_j(x_0, x)} : x \in S,\ f_i(x_0) > f_i(x),\ f_j(x_0) < f_j(x), 1 \le i, j \le p \right\}$$

is bounded from above.

Theorem 5. *Assume the same hypotheses as in Theorem 4. Then every efficient solution of (VOP) that satisfies the boundedness condition is properly efficient.*

3 Semilocally Preinvex and Related Functions in Pareto Optimization Problems

Another generalization of convexity, known as semilocal convexity, was introduced by Ewing [13] and investigated by Kaul and Kaur [24]. These authors give a new definition of generalized convexity by reducing the width of the segment path.

Definition 10. *A subset C of* $\mathbb{R}^n$ *is locally star-shaped at* $x_0 \in C$ *if for any* $x \in C$ *there exists a maximal positive number* $a(x, x_0) \le 1$ *such that*

$$\lambda x + (1-\lambda)x_0 \in C, \forall \lambda \in (0, a(x, x_0)).$$

A set $C \subseteq \mathbb{R}^n$ *is said to be locally star-shaped if it is locally star-shaped at each of its points.*

Note that each open set in $\mathbb{R}^n$ is a locally star-shaped set.

Definition 11. *Let C be a locally star-shaped set in* $\mathbb{R}^n$. *A scalar function* $f : C \to \mathbb{R}$ *is called semilocally convex on C if for any* $x, y \in C$ *there exists a positive number* $A(x, y) \le a(x, y)$ *such that*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y), \forall \lambda \in (0, A(x, y)).$$

A vector-valued function $f : C \to \mathbb{R}^p$ *is semilocally convex on C if each component is semilocally convex on C.*

Definition 12. *(Ewing [13]). A scalar function* $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ *is said to be semidifferentiable at* $x_0 \in C$ *if it is (one-sided) directionally differentiable at* x_0 *in the direction* $x - x_0$, *i.e, if*

$$Df(x_0, x - x_0) = \lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda(x - x_0)) - f(x_0)}{\lambda}$$

exists (finite or not) for each $x \in C$.

A vector-valued function $f : C \subseteq \mathbb{R}^n \to \mathbb{R}^p$ *is semidifferentiable at* x_0 *if each component of f is semidifferentiable at* x_0.

Ewing [13] proves that if a scalar function *f* is semilocally convex on the locally star-shaped set $C \subseteq \mathbb{R}^n$, then *f* is semidifferentiable on *C* and it holds that

$$f(x) - f(y) \geq Df(y, x - y), \forall x, y \in C.$$

Weir [38] has applied this concept to obtain a theorem of the alternative, optimality conditions and duality theorems for a nonsmooth constrained minimization problem. However, some results of his paper are not correct, as evidenced by Yang [40]. The same type of errors are subsumed in the paper of Mukherjee and Mishra [29]; see also Giorgi [16].

The previous definitions can be generalized in the following way (see also Giorgi [16], Preda et al. [33] and Preda [32]).

Definition 13. *The set C of* $\mathbb{R}^n$ *is* η*-locally star-shaped at* $x_0 \in C$ *if for any* $x \in C$ *there exists* $a(x, x_0) \in (0,1)$ *such that*

$$x_0 + \lambda \eta(x, x_0) \in C, \forall \lambda \in (0, a(x, x_0)).$$

Note that each open set in $\mathbb{R}^n$ is a η-locally star-shaped set at each of its points.

Definition 14. *Let* $f : C \to \mathbb{R}$, *where* $C \subseteq \mathbb{R}^n$ *is an* η*-locally star-shaped set at* $x_0 \in C$.

(i) We say that f is semilocally preinvex at x_0 *if, for each* $x \in C$, *there exists a positive number* $A(x, x_0) \leq a(x, x_0)$ *such that*

$$f(x_0 + \lambda \eta(x, x_0)) \leq \lambda f(x_0) + (1 - \lambda) f(x), \forall \lambda \in (0, A(x, x_0)).$$

(ii) We say that f is semilocally quasipreinvex at x_0 *if, for each* $x \in C$, *there exists a positive number* $A(x, x_0) \leq a(x_0, x)$ *such that*

$$f(x) \leq f(x_0), 0 < \lambda < A(x, x_0) \Rightarrow f(x_0 + \lambda \eta(x, x_0)) \leq f(x_0).$$

Definition 15. *Let* $f : C \to \mathbb{R}$ *de defined on the* η*-locally star-shaped set* $C \subseteq \mathbb{R}^n$ *at* $x_0 \in C$. *We say that f is* η*-semidifferentiable at* x_0 *if*

$$Df(x_0, \eta(x, x_0)) = \lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda \eta(x, x_0)) - f(x_0)}{\lambda}$$

exists (finite or not) for all $x \in C$.

Theorem 6. *Let* $f : C \to \mathbb{R}$ *be defined on the* η*-locally star-shaped set* $C \subseteq \mathbb{R}^n$. *If f is semilocally preinvex at* $x_0 \in C$, *then* $Df(x_0, \eta(x, x_0))$ *exists and we have that*

$$f(x) - f(x_0) \geq Df(x_0, \eta(x, x_0)), \forall x \in C.$$

Proof: Similar to the proof of Ewing [13].

Theorem 7. *Let C be an η-locally star-shaped set and let $f: C \to \mathbb{R}$ be an η-semidifferentiable function at $x_0 \in C$. If f is semilocally quasipreinvex at x_0, then*

$$f(x) \le f(x_0) \Rightarrow Df(x_0, \eta(x, x_0)) \le 0, \forall x \in C.$$

Proof: Given $x \in C$, as *f* is η-semidifferentiable at x_0, $Df(x_0, \eta(x, x_0))$ exists. Being *f* semilocally quasipreinvex at x_0, there exists a positive number $A(x, x_0) \le a(x, x_0)$ such that, $\forall \lambda \in (0, A(x, x_0))$ we have

$$f(x) \le f(x_0) \Rightarrow f(x_0 + \lambda\eta(x, x_0)) - f(x_0) \le 0.$$

Dividing both sides of the last inequality by $\lambda > 0$ and taking the limit for $\lambda \to 0^+$ we obtain the thesis.

We now take into consideration problem (VOP) in order to obtain sufficient optimality conditions. We assume that $X \in \mathbb{R}^n$ is a nonempty η-locally star-shaped set at $x_0 \in X$.

Lemma 1. *If for every $j \in J(x_0)$, $g_j : X \to \mathbb{R}$ is η-semilocally quasipreinvex at $x_0 \in S$ and for every $j \in J \setminus J(x_0)$, g_j is continuous at x_0, then the set S is η-locally star-shaped.*

Proof: First, let us observe that *X* is η-locally star-shaped at x_0 since is open, and so for each $x \in X$ there exists $a(x, x_0) \in (0,1)$ such that $x_0 + \lambda\eta(x, x_0) \in X$, $\forall \lambda \in (0, a(x, x_0))$. Given $x \in S$ one has $g_j(x) \le g_j(x_0) = 0$ for all $j \in J(x_0)$. As g_j is η-semilocally quasipreinvex at x_0, it follows that there exists a positive number $A_j(x, x_0) \le a(x, x_0)$ such that

$$g_j(x_0 + \lambda\eta(x, x_0)) \le g_j(x_0) = 0, \forall \lambda \in (0, A_j(x, x_0)), \forall j \in J(x_0). \tag{5}$$

For $j \in J \setminus J(x_0)$, $g_j(x_0) < 0$ and as g_j is continuous at x_0, there exists $A_j(x, x_0) > 0$ such that

$$g_j(x_0 + \lambda\eta(x, x_0)) \le 0 = g_j(x_0), \forall \lambda \in (0, A_j(x, x_0)), \forall j \in J \setminus J(x_0). \tag{6}$$

Choosing $b(x, x_0) = \min_{j \in J(x_0)} A_j(x, x_0)$, we have $b(x, x_0) \in (0,1)$ and conditions (5) and (6) imply that $x_0 + \lambda\eta(x, x_0) \in S$ for all $\lambda \in (0, b(x, x_0))$.

Theorem 8. *Consider problem (VOP) and assume the following:*

(a) For each $i \in I$, let f_i be η-semilocally preinvex at $x_0 \in S$.

(b) For each $j \in J(x_0)$ let g_j be η-semidifferentiable and η-semilocally quasipreinvex at $x_0 \in S$ and for each $j \in J \setminus J(x_0)$ let g_j be continuous at x_0.

Assume further that there exist $\lambda_i \ge 0$, $i \in I$, not all zero, and $\mu_j \ge 0$ for $j \in J(x_0)$ such that

$$\sum_{i=1}^{p} \lambda_i Df_i(x_0, \eta(x, x_0)) + \sum_{j \in J(x_0)} \mu_j Dg_j(x_0, \eta(x, x_0)) \ge 0, \forall x \in S. \tag{7}$$

Then x_0 is a weak minimum point for (VOP).

Proof: Assume that x_0 is not a weak minimum point for (VOP). Then there exists $x \in S$ such that $f_i(x) < f_i(x_0)$ for all $i \in I$. As $\lambda = (\lambda_i)_{i \in I} \ge 0$ is nonzero, we get

$$\lambda f(x) < \lambda f(x_0). \tag{8}$$

For $x \in S$ and for any $j \in J(x_0)$ we have $g_j(x) \le g_j(x_0) = 0$. By Lemma 1, the set S is η-locally star-shaped, and by Theorem 7 it follows that $Dg_j(x_0, \eta(x, x_0)) \le 0$. As $\mu_j \ge 0$, it holds that

$$\sum_{j \in J(x_0)} \mu_j Dg_j(x_0, \eta(x, x_0)) \le 0.$$

Using (7) we deduce that

$$\sum_{i=1}^{p} \lambda_i Df_i(x_0, \eta(x, x_0)) \ge 0.$$

But, being f_i η-semilocally preinvex at $x_0 \in S$, by Theorem 6 we obtain that $f_i(x) - f_i(x_0) \ge Df_i(x_0, \eta(x, x_0))$ for all $i \in I$, and therefore

$$\sum_{i=1}^{p} \lambda_i (f_i(x) - f_i(x_0)) \ge \sum_{i=1}^{p} \lambda_i Df_i(x_0, \eta(x, x_0)) \ge 0$$

in contradiction to (8).

Remark 1. (1) We can replace in Theorem 6 the hypotheses (a) by
(a') λf is η-semilocally preinvex at $x_0 \in S$,
and (7) by

$$D(\lambda f)(x_0, \eta(x, x_0)) + \sum_{j \in J(x_0)} \mu_j Dg_j(x_0, \eta(x, x_0)) \ge 0, \forall x \in S. \tag{9}$$

The proof is similar to the above.
(2) We can also replace the hypotheses (b) by
(b') $\mu g_{J(x_0)}$ is η-semidifferentiable and η-semilocally quasipreinvex at $x_0 \in S$ and for each $j \in J \setminus J(x_0)$, g_j is continuous at x_0, where $g_{J(x_0)} = (g_j)_{j \in J(x_0)}$
and (9) by

$$D(\lambda f)(x_0, \eta(x, x_0)) + D(\mu g_{J(x_0)})(x_0, \eta(x, x_0)) \ge 0, \forall x \in S.$$

But note that in this case, we do not have guarantee that S to be η-locally star-shaped.

4 Generalized Invexity and Local Cone Approximation

The strong growth of nonsmooth analysis, inspired above all by the works of Clarke [7], touched also the field of invex functions and their applications. Following Clarke's introduction of generalized directional derivatives and generalized subdifferentials for locally Lipschitz functions, it was natural to extend invexity to such functions. The main papers involved with nonsmooth invex functions, both in the sense of Clarke and following other treatments, are due to Craven [9], Craven and Glover [10], Giorgi and Guerraggio ([17], [18], [19]), Reiland ([34], [35]) and Kim and Schaible [26]. On the other hand, Elster and Thierfelder ([11], [12]) and independently Ward ([36], [37]) exploiting a general and axiomatic definition of local cone approximation of a set, introduced a general definition of directional derivative for a function $f: X \to \mathbb{R}$, where X is a finite dimensional space or also a topological linear space. See also Giorgi, Guerraggio and Thierfelder [20].

If $K(A, x_0) \subseteq \mathbb{R}^n$ is a local cone approximation at $x_0 \in \mathrm{cl}(A)$, where $A \subseteq \mathbb{R}^n$ and $\mathrm{cl}(A)$ denotes the closure of A, it is possible to approximate locally the set epi f (the epigraph of f) at the point $(x_0, f(x_0))$ by the cone K. So, a positively homogeneous function $f^K(x_0, \cdot)$ will be uniquely determined. More precisely, we have the following definition.

Definition 16. *Let $X \subseteq \mathbb{R}^n$ be an open set, let $f: X \to \mathbb{R}$, $x_0 \in X$ and $K(\cdot,\cdot)$ a local cone approximation. The positively homogeneous function $f^K(x_0, \cdot): \mathbb{R}^n \to [-\infty, \infty]$ defined by*

$$f^K(x_0, v) = \inf\{\beta \in \mathbb{R} : (v, \beta) \in K(\text{epi}\, f, (x_0, f(x_0)))\} \tag{10}$$

is called the K-directional derivative of f at x_0 in the direction $v \in \mathbb{R}^n$.

By means of Definition 16 we can recover most of the generalized directional derivatives used in the literature, for instance:

– The Dini upper directional derivative of f at x_0 in the direction v is

$$f^D(x_0, v) = \limsup_{t \to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

This derivative is associated to the cone of feasible directions.

$$F(A, x_0) = \{v \in \mathbb{R}^n : \forall (t_n) \to 0^+,\, x_0 + t_n v \in A\}.$$

– The Dini lower directional derivative of f at x_0 in the direction v

$$f_D(x_0, v) = \liminf_{t \to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t}$$

is associated to the cone of weak feasible directions

$$WF(A, x_0) = \{v \in \mathbb{R}^n : \exists (t_n) \to 0^+,\, x_0 + t_n v \in A\}.$$

– The Hadamard lower directional derivative of f at x_0 in the direction v

$$f_H(x_0, v) = \liminf_{t \to 0^+,\, v' \to v} \frac{f(x_0 + tv') - f(x_0)}{t}$$

is associated to the Bouligand tangent cone

$$T(A, x_0) = \{v \in \mathbb{R}^n : \exists (t_n) \to 0^+, \exists (v_n) \to v,\, x_0 + t_n v_n \in A\}.$$

– If f is locally Lipschitz, the Clarke directional derivative of f at x_0 in the direction v

$$f^0(x_0, v) = \limsup_{t \to 0^+,\, y \to x_0} \frac{f(y + tv) - f(y)}{t}$$

is associated to the Clarke tangent cone

$$TC(A,x_0)=\{v\in\mathbb{R}^n:\forall(x_n)\to x_0,\, x_n\in A,\, \forall(t_n)\to 0^+,\, \exists(v_n)\to v,\, x_n+t_nv_n\in A\}$$

Following these lines Castellani [5] proposes a unified definition of invexity for nonsmooth functions.

Definition 17. *Let* $K(\cdot,\cdot)$ *be a local cone approximation, the function* $f:X\to\mathbb{R}$ *is said to be:*

1) *K-invex if there exists a function* $\eta:X\times X\to\mathbb{R}^n$ *such that*

$$f(y)-f(x)\geq f^K(x,\eta(x,y)),\ \forall x,y\in X.$$

2) *K-pseudoinvex if there exists a function* $\eta:X\times X\to\mathbb{R}^n$ *such that*

$$f^K(x,\eta(x,y))\geq 0\Rightarrow f(y)\geq f(x),\ \forall x,y\in X.$$

3) *K-quasiinvex if there exists a function* $\eta:X\times X\to\mathbb{R}^n$ *such that*

$$f(y)\leq f(x)\Rightarrow f^K(x,\eta(x,y))\leq 0,\ \forall x,y\in X.$$

4) *Strictly K-quasiinvex if there exists a function* $\eta:X\times X\to\mathbb{R}^n$ *such that*

$$f(y)\leq f(x)\Rightarrow f^K(x,\eta(x,y))<0,\ \forall x\neq y,x,y\in X.$$

It is possible to prove that (similarly to the differentiable case) the class of *K*-invex functions coincides with the class of *K*-pseudoinvex functions.

Let us now consider problem (VOP). Extending to this problem the terminology of Castellani, we consider a feasible point x_0 for (VOP) and a local approximation $K(\cdot,\cdot)$. The point x_0 is said to be:

– a strongly efficient stationary point for problem (VOP), with respect to $K(\cdot,\cdot)$, if the following system is impossible in $v\in\mathbb{R}^n$:

$$(S_1)\begin{cases} f_i^K(x_0,v)<0, \text{ for some } i\in I\\ f_i^K(x_0,v)\leq 0, \text{ for all } i\in I\\ g_j^K(x_0,v)\leq 0, \text{ for all } j\in J(x_0).\end{cases}$$

– a weakly efficient stationary point for problem (VOP) with respect to $K(\cdot,\cdot)$, if the following system is impossible in $v\in\mathbb{R}^n$:

$$(S_2)\begin{cases} f_i^K(x_0,v)<0, \text{ for all } i\in I\\ g_j^K(x_0,v)<0, \text{ for all } j\in J(x_0).\end{cases}$$

It is always possible to choose a suitable local cone approximation $K(\cdot,\cdot)$ such that an efficient solution x_0 for (VOP) is a weakly or strongly efficient stationary point with respect to *K*. For instance, Antczak [2] proves that for $K=WF=F$ and $v=\eta(x_0,x)$ every efficient solution x_0 is a weakly efficient stationary point. Moreover, if some regularity condition holds, it is possible to prove that every weakly efficient stationary point is a strong stationary point.

It is possible, under suitable assumptions of K-invexity, to deduce sufficient optimality conditions directly from the impossibility of systems (S_1) or (S_2).

Theorem 9. *Let x_0 be a strongly stationary efficient point for problem (VOP) with respect to the cone $K(\cdot,\cdot)$. If every f_i is K-invex and every g_j, $j \in J(x_0)$, is K-quasiinvex (with respect to the same function η), then x_0 is an efficient solution for (VOP).*

Proof: Let x_0 be not efficient for (VOP), then there exists $x \in S$ such that

$$\begin{cases} f_i(x) < f_i(x_0), \text{ for some } i \in I \\ f_i(x) \leq f_i(x_0), \text{ for all } i \in I. \end{cases}$$

By K-invexity of $f_i, i \in I,$ we have

$$f_i^K(x_0, \eta(x_0, x)) < 0, \text{ for some } i \in I \tag{11}$$

$$f_i^K(x_0, \eta(x_0, x)) \leq 0, \text{ for all } i \in I. \tag{12}$$

Since $x_0 \in S$, then $g_j(x) \leq g_j(x_0) = 0$, for $j \in J(x_0)$. By K-quasiinvexity of g_j, $j \in J(x_0)$, we have

$$g_j^K(x_0, \eta(x_0, x)) \leq 0, \forall j \in J(x_0). \tag{13}$$

But (13), together with (11) and (12), contradicts the assumption that x_0 is a K-strongly stationary efficient point.

In order to prove the next theorem, we need a result of Weir and Mond [39].

Lemma 2. *Let X be a nonempty set in $\mathbb{R}^n$ and let $f: X \to \mathbb{R}^n$ be a preinvex function on X with respect to η (i.e. each of its components is preinvex with respect to η). Then either*

$$f(x) < 0 \textit{ has a solution } x \in X,$$

or

$$\lambda f(x) \geq 0, \forall x \in X, \textit{ for some non zero } \lambda \geq 0, \tag{14}$$

but both alternative are never true.

Theorem 10. *Let x_0 be a weakly stationary efficient point for problem (VOP) with respect to the cone $K(\cdot,\cdot)$. If every f_i, $i \in I$, is K-pseudoinvex, every g_j, $j \in J(x_0)$, is strictly K-quasiinvex (with respect to the same η) and every $f_i^K(x_0, \eta(x_0, \cdot))$, $i \in I$, is preinvex on X, then x_0 is a weak efficient solution for (VOP).*

Proof: For every $x \in S$, we have

$$g_j(x) \leq g_j(x_0) = 0, \forall j \in J(x_0)$$

and as g_j is strictly K-quasiinvex, it follows that

$$g_j^K(x_0, \eta(x_0, x)) < 0, \forall j \in J(x_0), \forall x \neq x_0.$$

But being x_0 a weakly stationary point, system (S_2) has no solution. So, we deduce that there is no solution $x \in S$ for the first components of system (S_2), i.e. the sub-system

$$f_i^K(x_0, \eta(x_0, x)) < 0, i \in I.$$

so, thanks to Lemma 2, there exist $\lambda_i \geq 0$, $i \in I$, not all zero, such that

$$\sum_{i \in I} \lambda_i f_i^K(x_0, \eta(x_0, x)) \geq 0, \forall x \in S. \tag{15}$$

Assume that x_0 is not a weak efficient solution for (VOP). Then there is a feasible point x of (VOP) such that $f_i(x) < f_i(x_0)$, $\forall i \in I$. As f is K-pseudoinvex, it follows that $f_i^K(x_0, \eta(x_0, x)) < 0$, $\forall i \in I$. From here, as $\lambda \neq 0$ we have $\sum_{i \in I} \lambda_i f_i^K(x_0, \eta(x_0, x)) < 0$, but this contradicts (15).

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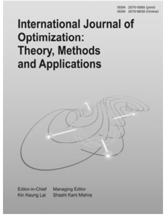

Resolvent Iterative Methods for Difference of Two Monotone Operators

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Abstract. In this paper, we consider the problem of finding a zero of difference of two monotone operators in Hilbert space. Using the resolvent operator technique, we establish the equivalence between the variational inclusions and the fixed point problems. This alternative equivalent formulation is used to suggest and analyze a two-step iterative method for solving the variational inclusions involving the difference of two monotone operators. We also show that the variational inclusions are equivalent to the resolvent equations. This equivalence formulation enables us to suggest and analyze a number of iterative methods. We consider the convergence analysis of these proposed iterative methods under suitable conditions. Our method of proofs are very simple as compared with other techniques. Several special cases are also discussed.

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1 Introduction

Variational inclusions involving the difference of two monotone operators provide us with a unified, natural, novel and simple framework to study a wide class of problems arising in DC programming, prox-regularity, multicommodity network, image restoring processing, tomograpy, molecular biology, optimization, pure and applied sciences, see [1-26] and the references therein. It is well known that the sum of two monotone operators is a monotone operator, whereas the difference of two monotone operators is not a monotone operator. Due to this reasons, the problem of finding a zero of the difference of two monotone operator is very difficult and has not been studied extensively. It is worth mentioning that this type of variational inclusions include as a special case the problem of finding the critical points of the difference of two convex functions. To the best of our knowledge, there does not exist a unified study of the variational inclusions involving the difference two monotone operators. Our present results are a contribution towards this goal. We would like to point out this problem has been considered by Adly and Oettli [1], Moudafi [7] and Noor et al [25] using quite different techniques. In this paper, we consider the problem of solving the variational inclusions involving the difference of two monotone operators. Using the resolvent operator technique, Noor et al [25] have shown that such type of variational inclusions are equivalent to the fixed point problem. This alternative formulation is used to suggest and analyze some two-step iterative methods for finding the zero of these variational inclusions. We also study the convergence of the new iterative method under some suitable conditions. These two-step methods include the one-step method considered by Noor et al [25] as a special case.

Related to the variational inclusions, we also consider the problem of finding the solving the resolvent equations associated with the difference of two monotone operators. We again use the resolvent operator technique to establish the equivalence between the resolvent equations and the variational inclusions. This alternative equivalent formulation is more flexible and unified. This equivalence has played an important part in suggesting some iterative methods for finding the zero of the difference of two(more) monotone operators. We also consider the convergence analysis of these iterative methods under suitable conditions. Our method of proofs of the results is very simple as compared with other methods. Some special cases are also discussed.

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$, respectively.

For given tow monotone operators $T, A : H \to H$, we consider the problem of finding $u \in H$ such that

$$0 \in A(u) - Tu. \tag{2.1}$$

Inequality of type (2.1) is called the variational inclusion involving the difference of two monotone operators. Note that the difference of two monotone operators is not a monotone operator as contrast to the sum of two monotone operators. Due to this reason, the problem of finding a zero of the difference of two monotone operators is very difficult to solve as compared to finding the zeroes of the sum of monotone operators. To the best of our knowledge, no research work has been carried out in this direction except that of Moudafi [7] and Noor et al [25]. See also Adly and Oettli [1].

If $A(.) \equiv \partial f(.)$, the subdifferential of a proper, convex and lower-semicontinuous function $f : H \longrightarrow R \cup \{\infty\}$, then problem (2.1) is equivalent to finding $u \in H$ such that

$$0 \in \partial f(u) - Tu, \tag{2.2}$$

a problem considered and studied by Adly and Oettli [1]. We note that problem (2.2) can be written as: find $u \in H$ such that

$$\langle Tu, v-u \rangle + f(u) - f(v) \leq 0, \quad \forall v \in H, \tag{2.3}$$

which is known as the mixed variational inequality or the variational inequality of the second kind. For the applications, numerical methods and other aspects of these mixed variational inequalities, see [1-28] and the references therein.

If f is the indicator function of a closed and convex set K in a real Hilbert space, then problem (2.3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u \rangle \leq 0, \quad \forall v \in K, \tag{2.4}$$

which is known as the classical variational inequality, introduced and studied by Stampacchia [27] in 1964. See also [1-28] for more details.

It is well known that the necessary optimality for the problem of finding the minimum of $f(x) - g(x)$, where $f(x)$ and $g(x)$ are differentiable convex functions, is equivalent to finding $x \in H$ such that

$$0 \in \left(\partial f(x) - \partial g(x)\right), \tag{2.5}$$

under some suitable conditions. Problem of type (2.5) have been considered in [3,5,6,7,28]. It is clear from the above discussion that problem (2.5) is a special case of problem (2.1). In fact, a wide class of problems arising in different branches of pure and applied sciences can be studied in the unified framework of problem (2.1). For appropriate and suitable choice of the operators and the space, one can obtain several new and known classes of variational inclusions, variational inequalities and complementarity problems, see [1-28] and the references therein.

We now recall some basic concepts and results.

Definition 2.1 [2]. If A is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \text{for all } u \in H,$$

where I is the identity operator.

It is well known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is a single-valued and nonexpansive, that is,

$$\| J_A(u) - J_A(v) \| \leq \| u - v \|, \quad \forall u, v \in H.$$

We now consider the problem of solving the resolvent equations. To be more precise, let $R_A = I - J_A$, where J_A is the resolvent operator associated with the maximal monotone operator A and I is the identity operator. For given nonlinear operators T, A, consider the problem of finding $z \in H$ such that

$$TJ_A z - \rho^{-1} R_A z = 0. \tag{2.6}$$

Equations of the type (2.6) are called the resolvent equations which were introduced and studied by Noor [9]. In particular, If $A(.) = \partial f(.)$, the subdifferential of a proper, convex and lower-semicontinuous function f, then it is well known that $J_A = P_K$, the projection of H onto the closed convex set K. In this case, resolvent equations are the Wiener-Hopf equations, which were introduced and studied by Shi [26] in connection with variational inequalities (2.4). This shows that the Wiener-Hopf equations are the special case of the resolvent equations. Resolvent equations technique has been used to study and develop several iterative methods for solving mixed variational inequalities and inclusions problems, see [9-26].

Definition 2.2. For all $u, v \in H$, an operator $T : H \to H$ is said to be:

(i) *strongly antimonotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v \rangle \leq -\alpha \| u - v \|^2$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\| Tu - Tv \| \leq \beta \| u - v \|.$$

We would like to point out that the differential $f'(.)$ of a strongly concave functions satisfies the Definition 2.1(i). Consequently, it is an antimonotone operator.

3 Resolvent Operator Method

In this section, we establish the equivalence between the variational inclusion (2.1) and the fixed point problem using the resolvent operator technique. This result is due to Noor et al [25]. This alternative formulation is used to discuss the existence of a solution of the problem (2.1) and to suggest and analyze an iterative method for solving the variational inclusions (2.1).

Lemma 3.1 [25]. Let A be a maximal monotone operator. Then, $u \in H$ is a solution of the variational inclusion (2.1), if and only if $u \in H$ satisfies the relation

$$u = J_A[u + \rho Tu], \tag{3.1}$$

where $J_A \equiv (I+\rho A)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

Proof: Let $u \in H$ be a solution of (2.1). Then

$$\begin{aligned} 0 &\in u+\rho A(u)-(\rho Tu+u)=(I+\rho A)(u)-(u+\rho Tu) \\ &\Longleftrightarrow \\ u &= (I+\rho A)^{-1}[u+\rho Tu]=J_A[u+\rho Tu], \end{aligned}$$

the required result.

Lemma 3.1 implies that the variational inclusion (2.1) is equivalent to the fixed point problem. This alternative equivalent formulation is very useful from the numerical and theoretical points of view. We rewrite the relation (3.1) in the following form

$$F(u)=J_A[u+\rho Tu], \tag{3.2}$$

which is used to study the existence of a solution of the variational inclusion (2.1).

We now study those conditions under which the variational inclusion (2.1) has a solution. This result is due to Noor et al [25]. We include its proof for the sake of completeness.

Theorem 3.1[25]. Let T be strongly antimonotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$. If there exist a constant ρ such that

$$0<\rho<\frac{2\alpha}{\beta^2}, \tag{3.3}$$

then there exists a solution of the problem (2.1).

Proof: From Lemma 3.1, it follows that problems (3.1) and (2.1) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (3.2), has a fixed point. For all $u \neq v \in H$, we have

$$\begin{aligned} \|F(u)-F(v)\| &= \|J_A[u+\rho Tu]-J_A[v+\rho Tv]\| \\ &\leq \|u-v+\rho(Tu-Tv)\|, \end{aligned} \tag{3.4}$$

where we have used the fact that the operator J_A is nonexpansive.

Since the operator T is strongly antimonotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned} \|u-v+\rho(Tu-Tv)\|^2 &\leq \|u-v\|^2 + 2\rho\langle Tu-Tv, u-v\rangle + \rho^2\|Tu-Tv\|^2 \\ &\leq (1-2\rho\alpha+\rho^2\beta^2)\|u-v\|^2 . \end{aligned} \tag{3.5}$$

From (3.5) and (3.4), we have

$$\|F(u)-F(v)\| \leq \sqrt{1-2\alpha\rho+\beta^2\rho^2}\,\|u-v\| = \theta\|u-v\|,$$

where

$$\theta=\sqrt{1-2\alpha\rho+\beta^2\rho^2}\,. \tag{3.6}$$

From (3.3), it follows that $\theta < 1,$ which implies that the map $F(u)$ defined by (3.2), has a fixed point, which is the unique solution of (2.1).

Using the technique of the updating the solution, we can rewrite the fixed point formulation (3.1) as:

$$w = J_A[u + \rho Tu] \tag{3.7}$$

$$u = J_A[w + \rho Tw], \tag{3.8}$$

which can be written as

$$u = J_A[J_A[u + \rho Tu] + \rho TJ_A[u + \rho Tu]].$$

These fixed point formulations are different from the fixed point formulation (3.1). We use these fixed point formulation to suggest the following two-step iterative methods for solving the variational inclusions (2.1) involving the difference of two monotone operators.

Algorithm 3.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$w_n = (1-\beta_n)u_n + \beta_n J_A[u_n + \rho Tu_n]$$
$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n J_A[w_n + \rho Tw_n], \quad n = 0,1,\cdots,$$

where $\alpha_n, \beta_n \in [0,1] \quad \forall n. \geq 0$. Algorithm 3.1 is known as a two-step iterative method for solving the variational inclusion (2.1).

For $\beta_n = 1,$ Algorithm 3.1 reduces to:

Algorithm 3.2. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$w_n = J_A[u_n + \rho Tu_n] \tag{3.9}$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n J_A[w_n + \rho Tw_n], \quad n = 0,1,\cdots, \tag{3.10}$$

where $\alpha_n \in [0,1] \quad \forall n. \geq 0.$ Algorithm 3.1 is also known as a two-step iterative method for solving the variational inclusion (2.1).

If $\beta_n = 0,$ then Algorithm 3.1 collapses to:

Algorithm 3.3. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n J_A[u_n + \rho Tu_n], \quad n = 0,1,\cdots,$$

which is known as the Mann iteration process for solving the variational inclusion (2.1). For the convergence analysis of Algorithm 3.3, see Noor et al [25].

If $A(.)$ is the indicator function of a closed convex set K in H, then $J_A = P_K$, the projection of H onto the closed convex set and consequently Algorithm 3.1 reduces to the following method.

Algorithm 3.4. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$w_n = (1-\beta_n)u_n + \beta_n P_K[u_n + \rho T u_n]$$
$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n P_K[w_n + \rho T w_n], \quad n = 0,1,\cdots,$$

where $\alpha_n, \beta_n \in [0,1] \quad \forall n. \geq 0$. Algorithm 3.1 is known as a two-step iterative method for solving the variational inclusion (2.1).

Algorithm 3.5. For a given $u_0 \in H$, find the approximate solution $u_{n=1}$ by the iterative scheme.

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n P_K[u_n + \rho T u_n], \quad n = 0,1,\cdots,$$

where $\alpha_n \in [0,1] \quad \forall n. \geq 0$.

We now consider the convergence analysis of Algorithm 3.2 and this is the main motivation of our next result. In a similar way, one can consider the convergence analysis of other Algorithms.

Theorem 3.2. Let the operator $T: H \longrightarrow H$ be strongly monotone with constants $\alpha > 0$ and Lipschitz continuous with constants with $\beta > 0$. If (3.3) holds and $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = = \infty$, then the approximate solution u_n obtained from Algorithm 3.2 converges to a solution $u \in H$ satisfying the variational inclusion (2.1).

Proof: Let $u \in H$ be a solution of the variational inclusion (2.1). Then, using Lemma 3.1, we have

$$u = (1-\alpha_n)u + \alpha_n J_A[u + \rho T u] \tag{3.11}$$

$$= J_A[u + \rho T u], \tag{3.12}$$

where $0 \leq \alpha_n \leq 1$ is a constant.

From (3.5), (3.6), (3.9) and (3.12), we have

$$\begin{aligned} \| w_n - u \| &\leq \| J_A[u_n + \rho T u_n] - J_A[u + \rho T u] \| \\ &\leq \| u_n - u + \rho(T u_n - T u) \| \\ &\leq \theta \| u_n - u \|. \end{aligned} \tag{3.13}$$

From (3.5), (3.6), (3.10), (3.11) and (3.13), we have

$$\begin{aligned} \| u_{n+1} - u \| &= \| (1-\alpha_n)(u_n - u) + \alpha_n \{J_A[w_n + \rho T w_n] - J_A[u + \rho T u]\} \| \\ &\leq (1-\alpha_n) \| u_n - u \| + \alpha_n \| w_n - u + \rho(T w_n - T u) \| \\ &\leq (1-\alpha_n) \| u_n - u \| + \alpha_n \sqrt{1 - 2\alpha\rho + \beta^2\rho^2} \| w_n - u \| \\ &\leq (1-\alpha_n) \| u_n - u \| + \alpha_n \theta^2 \| u_n - u \| \\ &= [1 - (1-\theta^2)\alpha_n] \| u_n - u \| \\ &\leq \prod_{i=0}^{n} [1 - (1-\theta^2)\alpha_i] \| u_0 - u \|. \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1-\theta^2 > 0$, we have $\lim_{n\to\infty} \{\prod_{i=0}^{n} [1-(1-\theta^2)\alpha_i]\} = 0$. Consequently the sequence $\{u_n\}$ convergences strongly to u. This completes the proof.

4 Resolvent Equations Technique

In this section, we first establish the equivalence between the variational inclusions (2.1) and the resolvent equations (2.6) using essentially the resolvent operator method. This equivalence is used to suggest and analyze some iterative methods for solving the variational inclusions.

Using Lemma 3.1, we show that the variational inclusions are equivalent to the resolvent equations.

Lemma 4.1. The variational inclusion (2.1) has a solution $u \in H$ if and only if the resolvent equations (2.6) have a solution $z \in H$, provided

$$u = J_A z \tag{4.1}$$

$$z = u + \rho T u, \tag{4.2}$$

where $\rho > 0$ is a constant.

Proof. Let $u \in H$ be a solution of (2.1). Then, from Lemma 3.1, we have

$$u = J_A[u + \rho T u]. \tag{4.3}$$

Taking $z = u + \rho T u$ in (4.3), we have

$$u = J_A z. \tag{4.4}$$

From (4.3) and (4.4), we have

$$z = u + \rho T u = J_A z + \rho T J_A z,$$

which shows that $z \in H$ is a solution of the resolvent equations (2.6). This completes the proof.

From Lemma 4.1, we conclude that the variational inclusion (2.1) and the resolvent equations (2.6) are equivalent. This alternative formulation plays an important and crucial part in suggesting and analyzing various iterative methods for solving variational inclusions and related optimization problems. In this paper, by suitable and appropriate rearrangement, we suggest a number of new iterative methods for solving variational inclusions (2.1).

I. The equations (2.6) can be written as

$$R_A z = \rho T J_A z,$$

which implies that, using(4.2)

$$z = J_A z + \rho T J_A z = u + \rho T u.$$

This fixed point formulation enables us to suggest the following iterative method for solving the variational inclusion(2.1).

Algorithm 4.1. For a given $z_0 \in H$, compute u_{n+1} by the iterative schemes

$$g(u_n) = J_A z_n \tag{4.5}$$

$$z_{n+1} = (1-\alpha_n)z_n + \alpha_n\{u_n + \rho Tu_n,\} \quad n = 0,1,2,\cdots, \tag{4.6}$$

where $0 \le \alpha_n \le 1$, for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

II. The equations (2.6) may be written as

$$\begin{aligned} z &= J_A z + \rho T J_A z + (1-\rho^{-1})R_A z \\ &= u + \rho Tu + (1-\rho^{-1})R_A z. \end{aligned}$$

Using this fixed point formulation, we suggest the following iterative method.

Algorithm 4.2. For a given $z_0 \in H$, compute u_{n+1} by the iterative schemes

$$\begin{aligned} u_n &= J_A z_n \\ z_{n+1} &= (1-\alpha_n)z_n + \alpha_n\{u_n + \rho Tu_n + (1-\rho^{-1})R_A z_n,\} \quad n=0, 1, 2, \cdots, \end{aligned}$$

where $0 \le \alpha_n \le 1$, for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

III. If the operator T is linear and T^{-1} exists, then the resolvent equation(2.6) can be written as

$$z = (I + \rho^{-1}T^{-1})R_A z,$$

which allows us to suggest the iterative method.

Algorithm 4.3. For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme

$$z_{n+1} = (1-\alpha_n)z_n + \alpha_n\{(I - \rho^{-1}T^{-1})R_A z_n,\} \quad n = 0,1,2\cdots,$$

where $0 \le \alpha_n \le 1$, for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

We would like to point out that one can obtain a number of iterative methods for solving the variational inclusion (2.1) for suitable and appropriate choices of the operators T, A and the space H. This shows that iterative methods suggested in this paper are more general and unifying ones.

We now study the convergence analysis of Algorithm 4.1. In a similar way, one can analyze the convergence analysis of other iterative methods.

Theorem 4.1. Let the operators T, A satisfy all the assumptions of Theorem 3.1. If the condition (3.3) holds, then the approximate solution $\{z_n\}$ obtained from Algorithm 4.1 converges to a solution $z \in H$ satisfying the Wiener-Hopf equation (2.6) strongly.

Proof. Let $u \in H$ be a solution of (2.1). Then, using Lemma 4.1, we have

$$z = (1-\alpha_n)z + \alpha_n\{u + \rho Tu\}, \tag{4.7}$$

where $0 \le \alpha_n \le 1$, and $\sum_{n=0}^{\infty} a_n = \infty$.

From(3.5), (4.6) and (4.7), we have

$$\begin{aligned} \| z_{n+1} - z \| &\le (1-\alpha_n) \| z_n - z \| + \alpha_n \| u_n - u + \rho(Tu_n - Tu) \| \\ &\le (1-\alpha_n) \| z_n - z \| + \alpha_n \left\{ \sqrt{1 - 2\rho\alpha + \beta^2} \right\} \| u_n - u \|, \end{aligned} \tag{4.8}$$

Also from (4.2), (4.5) and the nonexpansivity of the resolvent operator J_A, we have

$$\| u_n - u \| = \| J_A z_n - J_A z \| \le \| z_n - z \|. \tag{4.9}$$

Combining (4.8), and (4.9), we have

$$\| z_{n+1} - z \| \le (1-\alpha_n) \| z_n - z \| + \alpha_n \theta \| z_n - z \|, \tag{4.10}$$

where θ is defined by (3.6).

From (3.3), we see that $\theta < 1$ and consequently

$$\begin{aligned} \| z_{n+1} - z \| &\le (1-\alpha_n) \| z_n - z \| + \alpha_n \theta \| z_n - z \| \\ &= [1 - (1-\theta)\alpha_n] \| z_n - z \| \\ &\le \prod_{i=0}^{n} [1 - (1-\theta)\alpha_1] \| z_0 - z \|. \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1-\theta > 0$, we have $\lim_{n\to\infty} \prod_{i=0}^{n} [1-(1-\theta)\alpha_i] = 0$. Consequently the sequence $\{z_n\}$ convergences strongly to z in H, the required result.

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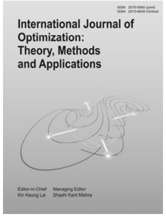

Solving Inverse Problems for Differential Equations by the Collage Method and Application to An Economic Growth Model

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Abstract. Inverse problems can be formulated for many mathematical problems and studied by different techniques; in this paper we analyze a technique based on the collage method for solving inverse problems arising in theory of DEs with initial conditions. Several numerical examples illustrate the use of this method for solving parameter identification problems. We present an economic model which involves the solution of an optimal control problem, and we show how one can apply the collage method to obtain estimates of parameter for this model.

Keywords: Collage Theorem, inverse problems, optimal control problems, economic growth.

1 Inverse Problems for Fixed Point Equations

We introduce a method of solving inverse problems for differential equations using fixed point theory for contractive operators. A number of inverse problems may be viewed in terms of the approximation of a target element x in a complete metric space (X, d) by the fixed point $\bar{x}$ of a contraction mapping $T: X \to X$. In practice, from a family of contraction mappings T_λ, $\lambda \in \Lambda \subset \mathbb{R}^n$, one wishes to find the parameter $\bar{\lambda}$ for which the approximation error $d(x, \bar{x}_\lambda)$ is as small as possible. Thanks to a simple consequence of Banach's fixed point theorem known as the "Collage Theorem", most

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practical methods of solving the inverse problem for fixed point equations seek to find an operator T for which the *collage distance* $d(x, Tx)$ is as small as possible.

Theorem 1. *("Collage Theorem" [1,2]) Let (X, d) be a complete metric space and* $T : X \to X$ *a contraction mapping with contraction factor* $c \in [0, 1)$*. Then for any* $x \in X$,

$$d(x, \bar{x}) \leq \frac{1}{1-c} d(x, Tx), \tag{1}$$

where $\bar{x}$ *is the fixed point of* T.

One now seeks a contraction mapping T_λ that minimizes the so-called *collage error* $d(x, T_\lambda x)$— in other words, a mapping that sends the target x as close as possible to itself. This is the essence of the method of *collage coding* which has been the basis of most, if not all, fractal image coding and compression methods [5]. Many problems in the parameter estimation literature for differential equations (see, e.g., [13]) can be formulated in such a collage coding framework as we showed in [9] and subsequent works [11,10,8,7,6].

2 Inverse Problem for DEs

Given a classical Cauchy problem for an ordinary differential equation

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0 \end{cases} \tag{2}$$

let us consider the Picard integral operator associated to it,

$$(Tx)(t) = x_0 + \int_0^t f(s, x(s))ds. \tag{3}$$

Suppose that f is Lipschitz in the variable x. Under these hypotheses T is Lipschitz on the space $C([-\delta, \delta]\times[-M, M])$ for some δ and $M > 0$. The following result states that the contractivity condition holds when the $\mathbb{L}^2$ distance is considered.

Theorem 2. *[9] The operator T satisfies*

$$\|Tu - Tv\|_2 \leq c\|u - v\|_2 \tag{4}$$

for all $u, v \in C([-\delta, \delta]\times[-M, M])$ *where* $c = \delta K$.

Now let $\delta' > 0$ be such that $\delta' K < 1$. Let $\{\varnothing_i\}$ be a basis of functions in $\mathbb{L}^2([-\delta', \delta')\times[-M, M])$ and consider the first n elements of this basis, that is,

$$f_a(s, x) = \sum_{i=1}^{n} a_i \varnothing_i(s, x). \tag{5}$$

Each vector of coefficients $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ then defines a Picard operator T_a. Suppose further that each function $\varnothing_i(s, x)$ is Lipschitz in x with constants K_i.

Theorem 3. *[4] Let* $\|K\|_2 = \left(\sum_{i=1}^n K_i^2\right)^{\frac{1}{2}}$ *and* $\|a\|_2 = \left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$. *Then*

$$|f_a(s, x_1) - f_a(s, x_2)| \le \|K\|_2 \|a\|_2 |x_1 - x_2| \tag{6}$$

for all $s \in [-\delta', \delta']$ *and* $x_1, x_2 \in [-M, M]$.

Given a target solution $x(t)$, we now seek to minimize the collage distance $\|x - T_a x\|_2$. The square of the collage distance becomes

$$\begin{aligned} \Delta(a)^2 &= \|x - T_a x\|_2^2 \\ &= \int_{-\delta}^{\delta} \left| x(t) - \int_0^t \sum_{i=1}^n a_i \varnothing_i(s, x(s)) ds \right|^2 dt \end{aligned} \tag{7}$$

and the inverse problem can be formulated as

$$\min_{a \in \mathbb{P}} \Delta(a), \tag{8}$$

where $\mathbb{P} = \left\{ a \in \mathbb{R}^n : \|K\|_2 \|a\|_2 < 1 \right\}$. The minimization may be performed by means of classical minimization methods.

3 Numerical Examples

Example 1. Consider the following system of differential equations:

$$\frac{dx_1}{dt} = x_2,\ x_1(0) = x_{10} \tag{9}$$

$$\frac{dx_2}{dt} = -bx_2 - kx_1,\ x_2(0) = x_{20}. \tag{10}$$

As an experiment simulation, we set $b = 1$, $k = 0.7$, $x_{10} = 0.1$ and $x_{20} = 0.5$. We solve numerically the system of ODEs. For $t \in [0, 30]$, we sample the solutions at 50 uniformly spaced points and degree 20 polynomials are fitted to the resulting simulated observational data. These two polynomials are our target functions. That is, we seek a Picard operator of the form

$$(Tx)(t) = x_0 + \int_0^t f(s, x(s)) ds$$

with

$$f(x) = \begin{pmatrix} c_1 x_2 \\ c_2 x_2 + c_3 x_1 \end{pmatrix},$$

and with the components of x_0 as parameters, as well. The result of the collage coding process is illustrated in Fig. 1. The minimal-collage system to five decimal places is

$$f(x) = \begin{pmatrix} 1.0006x_2 \\ -0.6980x_2 - 0.9954x_1 \end{pmatrix}, \; x_0 = \begin{pmatrix} 0.1020 \\ 0.4974 \end{pmatrix}.$$

Example 2. Let us consider the following system of random differential equations:

$$\begin{cases} \dfrac{dx_t}{dt} = Ax_t + B_t, \\ x_t \,|_{t=0} = x_0. \end{cases} \tag{11}$$

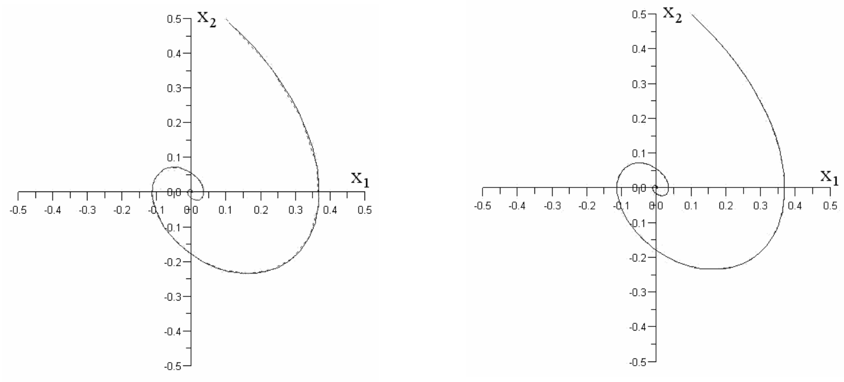

Fig. 1. Graphs in phase space. (left) the numerical solution (dashed) and the fitted target. (right) the target (dashed) and the fixed point of the resulting minimal-collage Picard operator.

where $x: \mathbb{R} \times \Omega \to \mathbb{R}^n$, A is a (deterministic) matrix of coefficients and B_t is a classical vector Brownian motion. An inverse problem for this kind of equation can be formulated as: given an i.d. sample of observations of $x(t, \omega)$, say $(x(t, \omega_1), \cdots, x(t, \omega_n))$, get an estimation of the matrix A. If we take the integral over Ω of both sides of the previous equation and recalling that $B_t \sim N(0, t)$, we have

$$\int_\Omega \frac{dx}{dt} dP = \frac{d}{dt} \mathbb{E}(x(t, \cdot)) = A\mathbb{E}(x(t, \cdot)) \tag{12}$$

This is a deterministic differential equation in $\mathbb{E}(x(t, \cdot))$ From the sample of observations of $x(t, \omega)$ we can get an estimation of $\mathbb{E}(x(t, \cdot))$ and then use the machinery of the previous section to solve the inverse problem for A. So the inverse problem for this system of random equations can be reduced to the analysis of a system of deterministic differential equations. As a numerical example, we consider the first-order system

$$\frac{d}{dt} x_t = a_1 x_t + a_2 y_t + b_t$$
$$\frac{d}{dt} y_t = b_1 x_t + b_2 y_t + c_t$$

Setting $a_1 = 0.5, a_2 = -0.4, b_1 = -0.3, b_2 = 1, x_0 = 0.9,$ and $y_0 = 1,$ we construct observational data values for x_t and y_t for $t_i = \frac{i}{N}, 1 \le i \le N,$ for various values of N. For each of M data sets, different pairs of Brownian motion are simulated for b_t and c_t. Fig. 2 presents several plots of b_t and c_t for $N = 100$. In Fig. 3, we present some plots of our generated x_t and y_t, as well as phase portraits for x_t versus y_t. For each sample time, we construct the mean of the observed data values, $x_{t_i}^*$ and $x_{t_i}^*$, $1 \le i \le N$. We minimize the squared collage distances

$$\Delta_x^2 = \frac{1}{N}\sum_{i=1}^{N}\left(x_{t_i}^* - x_0 - \frac{1}{N}\sum_{j=1}^{i}\left(a_1 x_{t_j}^* + a_2 x_{t_j}^*\right)\right)^2$$

and

$$\Delta_y^2 = \frac{1}{N}\sum_{i=1}^{N}\left(y_{t_i}^* - y_0 - \frac{1}{N}\sum_{j=1}^{i}\left(b_1 x_{t_j}^* + b_2 y_{t_j}^*\right)\right)^2$$

to determine the minimal collage parameters $a_1, a_2, b_1,$ and b_2. The results of the process are summarized in Table 1.

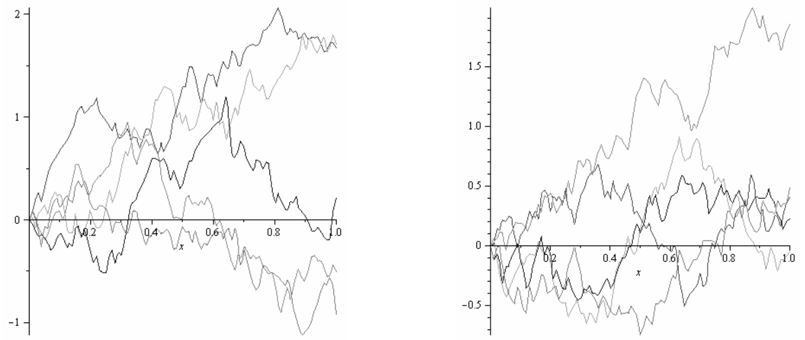

Fig. 2. Example plots of b_t and c_t for $N = 100$

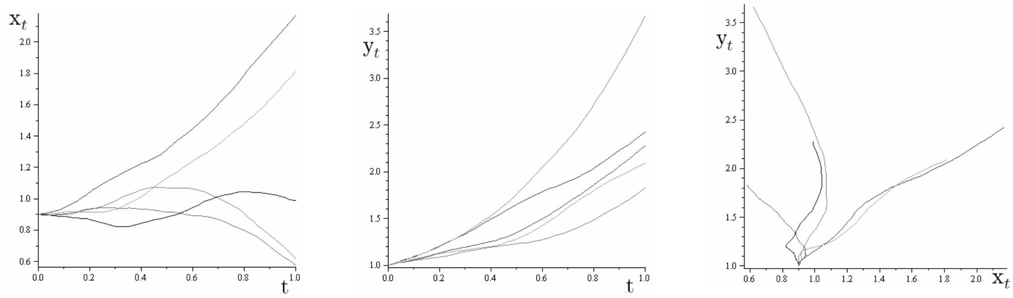

Fig. 3. Example plots of x_t, y_t, and x_t versus y_t for $N = 100$

Table 1. Minimal collage distance parameters for different N and M

N	M	a_1	a_2	b_1	b_2
100	100	0.2613	-0.2482	-0.2145	0.9490
100	200	0.3473	-0.3496	-0.2447	0.9700
100	300	0.3674	-0.3523	-0.2494	0.9462
200	100	0.3775	-0.3015	-0.1980	0.9252
200	200	0.3337	-0.3075	-0.2614	0.9791
200	300	0.4459	-0.3858	-0.2822	0.9718
300	100	0.4234	-0.3246	-0.2894	0.9838
300	200	0.3834	-0.3263	-0.3111	1.0000
300	300	0.5094	-0.4260	-0.3157	0.9965

Example 3. We now consider an inverse problem associated with the following variational problem:

$$\min_{x\in D} J(x) := \int_0^1 (x'(t))^2 + \alpha(t)x^2(t) + \beta(t)x(t)dt. \tag{13}$$

where $\alpha(t) \geq 0$ for all $t \in [0, 1]$ and the set of admissible functions for this problem is

$$D = \{x \in C^1([0,1]) \text{ s.t. } x(0) = 0\}. \tag{14}$$

It may be posed in the following way: Given a function $x^*(t)$, find J such that $x^*(t)$ is a solution of the variational problem. In this case, it is sufficient to find solutions of the following Euler equation with conditions,

$$2x''(t) - 2\alpha(t)x(t) - \beta(t) = 0, \quad x(0) = 0, \quad x'(1) = 0. \tag{15}$$

(see [14].) Integrating this equation twice yields

$$x'(t) - x'(0) = \int_0^t \alpha(s)x(s)ds + \frac{1}{2}\int_0^t \beta(s)ds \tag{16}$$

and

$$x(t) = x'(0)t + \int_0^t \left(\int_0^r \alpha(s)x(s)ds + \frac{1}{2}\int_0^t \beta(s)ds \right) dr. \tag{17}$$

From Eq. (16) and the condition $x'(1) = 0$, we have

$$x'(0) = -\int_0^1 \alpha(s)x(s)ds + \frac{1}{2}\int_0^1 \beta(s)ds \tag{18}$$

so that the second order differential equation can be reduced to the following fixed point equation

$$
\begin{aligned}
x(t) &= T(x)(t) \\
&= -\left(\int_0^1 \alpha(s)x(s)dx + \frac{1}{2}\int_0^1 \beta(s)ds \right) t \\
&\quad + \int_0^t \left(\int_0^r \alpha(s)x(s)ds + \frac{1}{2}\int_0^t \beta(s)ds \right) dr,
\end{aligned}
\tag{19}
$$

with the initial condition $x(0) = 0$. So the inverse problem for this variational optimization problem can be reduced to an inverse problem for a differential equation with initial conditions.

4 An Application: Parameter Identification for An Economic Growth Model

We now show how one can use the previous method for obtaining estimates of parameters for an optimal control problem arising in economic growth. In practical applications, one seeks to obtain such estimates of meaningful parameters starting from a set of observational data. As an example, let us consider the following problem. It consists of a representative infinitely-lived household seeking to maximize under constraints its lifetime discounted utility:

$$
\max_{C(t)} \int_0^{+\infty} \ln(c(t))L(t)e^{-pt}dt \tag{20}
$$

subject to:

$$
\begin{cases}
\dot{K}(t) = Y(t) - \beta_K K(t) - C(t) - G(t) \\
\dot{A}(t) = g_A A(t) \\
\dot{L}(t) = nL(t) - dL^2(t) \\
\dfrac{d}{dt}\left(\dfrac{G(t)}{Y(t)}\right) = \mu\left(\dfrac{G(t)}{Y(t)}\right) - \gamma\left(\dfrac{G(t)}{Y(t)}\right)^2
\end{cases}
$$

where $c(t) \equiv \dfrac{C(t)}{L(t)}$ and $K(t)$ is the physical capital, $A(t)$ is the technological progress, $L(t)$ is the population, $G(t)$ is the public expenditure, and $Y(t) = A^{\eta}(t)K^{\alpha}(t)L^{\theta}(t)G^{1-\eta-\alpha-\theta}(t)$. Let $K(0) = K_0$, $A(0) = A_0$, $L(0) = L_0$ and $G(0) = G_0$, be the initial conditions. This model has been introduced by Bucci, Florio and La Torre in [3] where one can find more details and motivations about it. Here we are interested in analyzing the inverse problem for this model. All differential equations can be easily solved in a closed form. We get:

$$
\begin{aligned}
A(t) &= A_0 e^{gAt} \\
L(t) &= \frac{n}{d + \left(\dfrac{n}{L_0} - d\right)e^{-nt}} \\
\frac{G(t)}{Y(t)} &= \frac{\mu}{\gamma + \left(\dfrac{\mu Y_0}{G_0} - \gamma\right)e^{-\mu t}}
\end{aligned}
$$

Upon substituting all these expressions, by easy calculations we can reduce the problem to the following:

$$\max_{C(t)} \int_0^{+\infty} \ln(C(t))L(t)e^{-pt}dt \tag{21}$$

subject to

$$\dot{K}(t) = \left(1 - \frac{\mu}{\gamma + \left(\frac{\mu Y_O}{G_O} - \gamma\right)e^{-\mu t}}\right)\left(\frac{\mu}{\gamma + \left(\frac{\mu Y_O}{G_O} - \gamma\right)e^{-\mu t}}\right)^{\frac{1-\alpha-\theta-\eta}{\alpha+\theta+\eta}} A(t)^{\frac{\eta}{\alpha+\theta+\eta}} K(t)^{\frac{\alpha}{\alpha+\theta+\eta}} L(t)^{\frac{\theta}{\alpha+\theta+\eta}} - \beta_K K(t) - C(t)$$

By defining

$$\xi(t) = \left(1 - \frac{\mu}{\gamma + \left(\frac{\mu Y_O}{G_O} - \gamma\right)e^{-\mu t}}\right)\left(\frac{\mu}{\gamma + \left(\frac{\mu Y_O}{G_O} - \gamma\right)e^{-\mu t}}\right)^{\frac{1-\alpha-\theta-\eta}{\alpha+\theta+\eta}} \tag{22}$$

the first order conditions (FOCs) can be reduced to the following nonlinear system:

$$\begin{cases} \dfrac{\dot{C}(t)}{C(t)} = \dfrac{\dot{L}(t)}{L(t)} - \rho + \dfrac{\alpha}{\alpha+\theta+\eta}\xi(t)A(t)^{\frac{\eta}{\alpha+\theta+\eta}} K(t)^{\frac{\alpha}{\alpha+\theta+\eta}-1} L(t)^{\frac{\theta}{\alpha+\theta+\eta}} - \beta_K \\ \dfrac{\dot{K}(t)}{K(t)} = -\beta_K - \dfrac{C(t)}{K(t)} + \xi(t)A(t)^{\frac{\eta}{\alpha+\theta+\eta}} K(t)^{\frac{\alpha}{\alpha+\theta+\eta}-1} L(t)^{\frac{\theta}{\alpha+\theta+\eta}} \end{cases}$$

To simulate the solution of this system, let us consider the following values of the unknown parame ters: $\alpha = 0.36, \eta = 0.21, \theta = 0.19, g_A = 6.4\%, n = 0.0144, d = 2.28571428610^{-7}, \rho = 0.01, \frac{G_O}{Y_O} = 20\%$, $\beta_K = 0.001, \mu = 0.5, \gamma = 1$ and the following initial conditions: $L(0) = 0.01439999670$, $A(0) = 1$, $G(0) = 4$, $K(0) = 10^{3.33}$, by $C_0 = 6.372639689$. For motivation about the choice of this set of values one can see [3]. The behavior of $C(t)$ and $K(t)$ are shown in Fig. 4.

As an example of an inverse problem, we suppose that we know the values of α, η, and θ, and that we can gather observational data for $C(t)$, $K(t)$, $A(t)$, $L(t)$, $G(t)$, and $Y(t)$. From this information, we seek to recover estimates of the remaining parameter values. On an observational interval of length 40, we gather 80 uniformly spaced data values for each of our functions. We perform a least-squares fit of a sixth degree polynomial to each of these data sets, producing our approximation of each of the six functions, which we refer to as our "target functions". To solve the inverse problem, we seek values of the parameters bi so that the system of DEs of the form

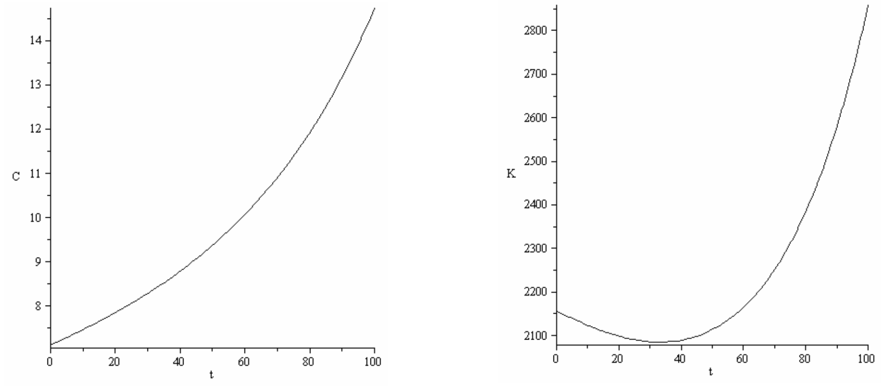

Fig. 4. (left to right) $C(t)$ and K(t)

$$\dot{K} = Y + b_1 K - C - G \tag{23}$$

$$\dot{A} = b_2 A \tag{24}$$

$$\dot{L} = b_3 L + b_4 L^2 \tag{25}$$

$$\left(\frac{\dot{G}}{Y}\right) = b_5 \frac{G}{Y} + b_6 \left(\frac{G}{Y}\right)^2 \tag{26}$$

$$\dot{C} = \left(\frac{\dot{L}}{L} + b_7 + \frac{\alpha}{\alpha+\theta+\eta}\left(\frac{\dot{K}}{K} + b_1 + \frac{C}{K}\right) - b_1\right)C \tag{27}$$

admits the corresponding target functions as an approximate solution. Our approach is to minimize the squared $\mathbb{L}^2$ collage distance componentwise. We first treat components (23)-(26), recovering estimates of $b_i, i = 1, \cdots, 6$. Then we treat component (27) to obtain a value for b_7. To eight decimal places, the results along with desired true values of the coefficients are presented in Table 2.

Table 2. Results for the economic growth model inverse problem

parameter	collage-coded value	true value
b_1	-0.00099134	$-\beta_K = -0.001$
b_2	0.06399086	$g_A = 0.064$
b_3	0.01439999	$n = 0.0144$
b_4	-0.00000023	$-d = -0.00000023$
b_5	0.54443991	$\mu = 0.5$
b_6	-1.09100872	$-\gamma = -1$
b_7	-0.01104799	$-\rho = -0.01$

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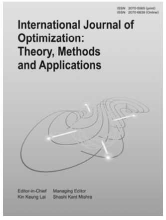

Some (h, φ)-differentiable Multi-objective Programming Problems

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Abstract. In the setting of Ben-Tal's generalized algebraic operations, we deal with a multi-objective programming problem where all functions involved are (h, φ)-differentiable. We introduce new generalized classes of type I functions, called (h, φ)-(F, α, ρ, d)-type I, quasi (h, φ)-(F, α, ρ, d)-type I, and pseudo (h, φ)-(F, α, ρ, d)-type I, by combing the notions of (h, φ)-differentiability and (F, α, ρ, d)-convexity. Based upon these generalized type I functions, we obtain a few sufficient optimality conditions and prove some results on duality.

Keywords: multi-objective programming, weak Pareto solution, sufficiency, type I functions, generalized convexity, duality.

1 Introduction

It is well known that convexity plays an vital role in deriving sufficient optimality conditions and duality results in multi-objective programming problem. To relax convexity assumptions imposed on the functions in theorems on sufficient optimality and duality, various generalized convexity notions have been presented in the literature [1-2,5-26]. Among them, we recall two important concepts, one is type I functions which was given by Hanson and Mond [9-10]. With and without differentiability, the type I functions were extended and applied to establish the optimality conditions and dual theory for different mathematical programming problems by some researchers. For

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example: Aghezzaf and Hachimi [1], Gulati and Agarwal [5], Hachimi and Aghezzaf [6-7], Kaul et al. [12], Mishra [14], Mishra and Noor [16], Mishra et al. [15,17-18], Rueda and Hanson [20], Rueda et al. [21] and Suneja and Srivastava [24] etc. The second one is (F, α, ρ, d)-convexity, which was introduced by Liang et al. [13] as extension of several concepts of generalized convexity. For example: F-convexity [4], ρ -convexity[25] and (F, ρ)-convexity [2,19].

In the literature [3-4], Ben-Tal introduced certain generalized operations of addition and multiplication. With the help of Ben-Tal's generalized algebraic operations, a meaningful generalization of convex functions is the introduction of (h,φ)-convex functions, which was given in Avriel [3]. In addition, the concept of (h,φ)-differentiability was introduced in the literature [3]. Recently, Xu and Liu [27], established Kuhn-Tucker necessary optimality conditions for a multi-objective optimization, where all functions involved are (h,φ)-differentiable.

In present paper, we introduce new classes of type I and generalized type I functions, called (h,φ)-(F, α, ρ, d)-type I and generalized (h,φ)-(F, α, ρ, d)-type I, by combining the concepts of (F, α, ρ, d)-convexity [13] and type I [10] and generalized type I function [12,20-21], for a multi-objective differentiable programming problem in the setting of Ben-Tal's generalized algebraic operations. We derive some Karush-Kuhn-Tucker type of sufficient optimality conditions and duality theorems for a weak Pareto solution to the problem involving the new classes of type I and generalized type I functions. This paper is divided into four sections. Section 2 includes preliminaries and related results which will be used in later sections. Section 3 and Section 4 are devoted to establishing sufficient conditions of optimality and duality theorems, respectively.

2 Preliminaries and Related Results

Let $\mathbb{R}^n$ Rn be the n-dimensional Euclidean space, and $\mathbb{R}, \mathbb{R}_+$ be the sets of all real numbers and nonnegative numbers, respectively. Throughout this paper, the following convention for vectors in $\mathbb{R}^n$ will be followed:

$$\begin{aligned}
&x < y \quad \text{if and only if} \quad x_i < y_i,\ i = 1, 2, \cdots, n,\\
&x \leqq y \quad \text{if and only if} \quad x_i \leqq y_i,\ i = 1, 2, \cdots, n,\\
&x \leq y \quad \text{if and only if} \quad x_i \leqq y_i,\ i = 1, 2, \cdots, n, \text{ but } x \neq y,\\
&x \not< y \quad \text{is the negation of } x < y.
\end{aligned}$$

Now, let us recall generalized operations of addition and multiplication introduced by Ben-Tal in [3].

1) Let h be an n vector-valued continuous function, defined on $\mathbb{R}^n$ and possessing an inverse function h^{-1}. Define the h-vector addition of $x, y \in \mathbb{R}^n$ as

$$x \oplus y = h^{-1}(h(x) + h(y)), \tag{1}$$

and the h-scalar multiplication of $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ as

$$\alpha \otimes x = h^{-1}(\alpha h(x)). \tag{2}$$

2) Let φ be real-valued continuous function, defined on $\mathbb{R}$ and possessing an inverse function φ^{-1}. Then the φ-addition of two numbers, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, is given by

$$\alpha[+]\beta = \varphi^{-1}(\varphi(\alpha) + \varphi(\beta)), \tag{3}$$

and the φ-scalar multiplication of $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ as

$$\beta[\cdot]\alpha = \varphi^{-1}(\beta\varphi(\alpha)). \tag{4}$$

Denote

$$\bigoplus_{i=1}^{m} x^i = x^1 \oplus x^2 \oplus \cdots \oplus x^m,\ x^i \in \mathbb{R}^n,\ i = 1, 2, \cdots, m, \tag{5}$$

$$[\sum_{i=1}^{m}]\alpha_i = \alpha_1[+]\alpha_2[+]\cdots[+]\alpha_m,\ \alpha_i \in \mathbb{R},\ i = 1, 2, \cdots, m, \tag{6}$$

$$\alpha[-]\beta = \alpha[+]((-1)[\cdot]\beta),\ \alpha,\ \beta \in \mathbb{R}. \tag{7}$$

In the above Ben-Tal generalized algebraic operations, it is worth noting that $\beta[\cdot]\alpha$ may not be equal to $\alpha[\cdot]\beta$ for $\alpha,\ \beta \in \mathbb{R}$. In addition, it is clear that $1 \otimes x = x$ for any $x \in \mathbb{R}^n$ and $1[\cdot]\alpha = \alpha$ for any $\alpha \in \mathbb{R}$. For $\alpha,\ \beta \in \mathbb{R}$. and $x \in \mathbb{R}^n$, the following conclusions can be obtained with easy

$$\begin{aligned} \varphi(\alpha[\cdot]\beta) = \alpha\varphi(\beta),\ h(\alpha \otimes x) = \alpha h(x), \\ \alpha[-]\beta = \varphi^{-1}(\varphi(\alpha) - \varphi(\beta)). \end{aligned} \tag{8}$$

Avriel [3] introduced the following concept, which plays an important role in our paper.

Definition 2.1 Let f be a real-valued function defined on $\mathbb{R}^n$, denote $\hat{f}(t) = \varphi(f(h^{-1}(t))),\ t \in \mathbb{R}^n$. For simplicity, write $\hat{f}(t) = \varphi f h^{-1}(t)$. The function f is said to be (h, φ)-differentiable at $x \in \mathbb{R}^n$, if $\hat{f}(t)$ is differentiable at $t = h(x)$, and denoted by $\nabla^* f(x) = h^{-1}(\nabla \hat{f}(t)\big|_{t=h(x)})$. In addition, It is said that f is (h, φ)-differentiable on $X \subset \mathbb{R}^n$ if it is (h, φ)-differentiable at each $x \in X$. A vector-valued function is called (h, φ)-differentiable on $X \subset \mathbb{R}^n$ if each of its components is (h, φ)-differentiable on X.

If f is differentiable at x, then f is (h, φ)-differentiable at x. We obtain this fact by setting h and φ are identity functions, respectively. However, the converse is not true. For example, let $f(x) = \sqrt{|x-1|}$ be a function defined on $\mathbb{R}$. It is clear that f is not differentiable at $x = 1$, but f is (h, φ)-differentiable at $x = 1$, where $h(x) = x,\ \varphi(x) = x^3,\ x \in \mathbb{R}$.

Definition 2.2 Let X be a nonempty subset of $\mathbb{R}^n$. A functional F: $X \times X \times \mathbb{R}^n \to \mathbb{R}$ is called (h, φ)-sublinear if for any $x, \bar{x} \in X$,

$$F(x, \bar{x}; a_1 \oplus a_2) \leqq F(x, \bar{x}; a_1)[+]F(x, \bar{x}; a_2)\ \forall\ a_1, a_2 \in \mathbb{R}^n,$$

$$F(x,\bar{x};\alpha\otimes\alpha)\leqq\alpha[\cdot]F(x,\bar{x};a)\quad\forall a\in\mathbb{R}^n,\alpha\geqq 0,\forall a\in\mathbb{R}^n.$$

From the above definition, we can easy obtain that if F is a (h,φ)-sublinear functional then

$$F\left(x,\bar{x};\overset{m}{\underset{i=1}{\oplus}}a_i\right)\leqq[\sum_{i=1}^{m}]F(x,\bar{x};a_i)\,a_i\in\mathbb{R}^n\ i=1,\cdots,m. \tag{9}$$

We collect the following properties of Ben-Tal generalized algebraic opera-tions and (h,φ)-differentiable functions from literature [27], which will be used in the squeal.

Lemma 2.1 Suppose that f is a real-valued function defined on $\mathbb{R}^n$, and (h,φ)-differentiable at $\bar{x}\in\mathbb{R}^n$. Then, the following statements hold:

(a) Let $x^i\in\mathbb{R}^n,\lambda_i\in\mathbb{R},i=1,2,\cdots,m.$ Then

$$\overset{m}{\underset{i=1}{\oplus}}(\lambda_i\otimes x^i)=h^{-1}\left(\sum_{i=1}^{m}\lambda_i h(x^i)\right),\overset{m}{\underset{i=1}{\oplus}}x^i=h^{-1}\left(\sum_{i=1}^{m}h(x^i)\right).$$

(b) Let $\mu_i,\alpha_i\in\mathbb{R},i=1,2,\cdots,m.$ Then

$$[\sum_{i=1}^{m}](\mu_i[\cdot]\alpha_i)=\varphi^{-1}\left(\sum_{i=1}^{m}\mu_i\varphi(\alpha_i)\right),\ [\sum_{i=1}^{m}]\alpha_i=\varphi^{-1}\left(\sum_{i=1}^{m}\varphi(\alpha_i)\right)$$

(c) For $\alpha\in\mathbb{R},\alpha[\cdot]f$ is (h,φ)-differentiable at $\bar{x}$ and $\nabla^*(\alpha[\cdot]f(\bar{x}))=\alpha\otimes\nabla^*f(\bar{x})$.

We need more properties of Ben-Tal generalized algebraic operations.

Lemma 2.2 Let $i=1,2,\cdots,m.$ The following statements hold:

(a) For $\alpha,\beta,\gamma\in\mathbb{R}$, then $\alpha[\cdot](\beta[\cdot]\gamma)=\beta[\cdot](\alpha[\cdot]\gamma)=(\alpha\beta)[\cdot]r.$

(b) For $\beta,\alpha_i,\in\mathbb{R}$, then $\beta[\cdot][\sum_{i=1}^{m}]\alpha_i=[\sum_{i=1}^{m}](\beta[\cdot]\alpha_i).$

(c) For $\alpha,\beta,\gamma\in\mathbb{R}$, then $\gamma[\cdot](\alpha[-]\beta)=(\gamma[\cdot]\alpha)[-](\gamma[\cdot]\beta).$

(d) For $\alpha_i,\beta_i\in\mathbb{R}$, then

$$[\sum_{i=1}^{m}](\alpha_i[-]\beta_i)=[\sum_{i=1}^{m}]\alpha_i[-][\sum_{i=1}^{m}]\beta_i,\ \ [\sum_{i=1}^{m}]\alpha_i(\alpha_i[+]\beta_i)=[\sum_{i=1}^{m}]\alpha_i[+][\sum_{i=1}^{m}]\beta_i.$$

Proof:

(a) It is easy to obtain this fact from (4).

(b) We can get from Lemma 2.1 (b) and (4) and that

$$\begin{aligned}[\sum_{i=1}^{m}](\beta_i[\cdot]\alpha_i)&=\varphi^{-1}\left(\beta\sum_{i=1}^{m}\varphi(a_i)\right)\\&=\varphi^{-1}\left(\beta\varphi(\varphi^{-1}(\sum_{i=1}^{m}\varphi(\alpha_i)))\right)\\&=\beta[\cdot]\varphi^{-1}\left(\sum_{i=1}^{m}\varphi(\alpha_i)\right)\\&=\beta[\cdot][\sum_{i=1}^{m}]\,\alpha_i.\end{aligned}$$

(c) It follows form (7) and Lemma 2.2 (a-b) that

$$\begin{aligned}\gamma[\cdot](\alpha[-]\beta) &= \gamma[\cdot](\alpha[+](-1)[\cdot]\beta) \\ &= (\gamma[\cdot]\alpha)[+](-1)[\cdot](\gamma[\cdot]\beta) \\ &= (\gamma[\cdot]\alpha)[-](\gamma[\cdot]\beta).\end{aligned}$$

(d) By Lemma 2.1 (b) and (8), We can see that

$$\begin{aligned}[\sum_{i=1}^{m}](\alpha_i[-]\beta_i) &= [\sum_{i=1}^{m}]\varphi^{-1}(\varphi(\alpha_i)-\varphi(\beta_i)) \\ &= \varphi^{-1}\left(\sum_{i=1}^{m}\varphi(\varphi^{-1}(\varphi(\alpha_i)-\varphi(\beta_i)))\right) \\ &= \varphi^{-1}\left(\sum_{i=1}^{m}\varphi(\alpha_i)-\sum_{i=1}^{m}\varphi(\alpha_i)\right) \\ &= \varphi^{-1}\left(\varphi(\varphi^{-1}(\sum_{i=1}^{m}\varphi(\alpha_i)))-\varphi(\varphi^{-1}(\sum_{i=1}^{m}\varphi(\alpha_i)))\right) \\ &= \varphi^{-1}\left(\varphi([\sum_{i=1}^{m}]\alpha_i)-\varphi([\sum_{i=1}^{m}]\beta_i)\right) \\ &= [\sum_{i=1}^{m}]\alpha_i[-][\sum_{i=1}^{m}]\beta_i.\end{aligned}$$

With the similar arguments, we can obtain the second equation. The proof is completed.

Lemma 2.3 Suppose that function φ, appears in Ben-Tal generalized algebraic operations, is strictly monotone with $\varphi(0)=0$. Then, the following statements hold:

(a) Let $\gamma \geqq 0$, $\gamma, \alpha, \beta \in \mathbb{R}$, and $\alpha \leqq \beta$. Then $\gamma[\cdot]\alpha \leqq \gamma[\cdot]\beta$.
(b) Let $\gamma \geqq 0$, $\gamma, \alpha, \beta \in \mathbb{R}$, and $\alpha < \beta$. Then $\gamma[\cdot]\alpha \leqq \gamma[\cdot]\beta$.
(c) Let $\gamma > 0$, $\gamma, \alpha, \beta \in \mathbb{R}$, and $\alpha < \beta$. Then $\gamma[\cdot]\alpha < \gamma[\cdot]\beta$.
(d) Let $\gamma < 0$, $\gamma, \alpha, \beta \in \mathbb{R}$, and $\alpha \geqq \beta$. Then $\gamma[\cdot]\alpha \leqq \gamma[\cdot]\beta$.
(e) Let $\alpha_i, \beta_i \in \mathbb{R}, i \in M = \{1, 2, \cdots, m\}$. if $\alpha_i \leqq \beta_i$ for any $i \in M$, then

$$[\sum_{i=1}^{m}]\alpha_i \leqq [\sum_{i=1}^{m}]\beta_i; \tag{10}$$

If $\alpha_i \leqq \beta_i$ for any $i \in M$ and there exists at least an index $k \in M$ such that $\alpha_k < \beta_k$, then

$$[\sum_{i=1}^{m}]\alpha_i < [\sum_{i=1}^{m}]\beta_i. \tag{11}$$

Proof: For conclusions (a) − (d), we only prove (a), because the proof of (b) − (d) is similar to that of (a). Without loss of generality, we suppose that φ is strictly monotone increasing on $\mathbb{R}$.
(a) Since $\gamma \geqq 0$, we have that

$$\begin{aligned}\alpha \leqq \beta \Rightarrow \varphi(\alpha) \leqq \varphi(\beta) \Rightarrow \gamma\varphi(\alpha) \leqq \gamma\varphi(\beta) \\ \Rightarrow \varphi^{-1}(\gamma\varphi(\alpha)) \leqq \varphi^{-1}(\gamma\varphi(\beta)) \\ \Leftrightarrow \gamma[\cdot]\alpha \leqq \gamma[\cdot]\beta.\end{aligned}$$

(e) We only prove (11), (10) can be obtained in the similar way. By the given conditions, we have

$$\alpha_k < \beta_k, \quad \alpha_i \leqq \beta_i \quad \text{for all } i \neq k.$$

Hence,

$$\varphi(\alpha_k) < \varphi(\beta_k), \quad \varphi(\alpha_i) \leqq \varphi(\beta_i) \quad \text{for all } i \neq k.$$

Consequently,

$$\sum_{i=1}^{m} \varphi(\alpha_i) < \sum_{i=1}^{m} \varphi(\beta_i).$$

Since φ is strictly monotone increasing, we get

$$\varphi^{-1}\left(\sum_{i=1}^{m} \varphi(\alpha_i)\right) < \varphi^{-1}\left(\sum_{i=1}^{m} \varphi(\beta_i)\right).$$

It yields from Lemma 2.1 (b) that $\left[\sum_{i=1}^{m}\right]\alpha_i < \left[\sum_{i=1}^{m}\right]\beta_i$.

Lemma 2.4 Suppose that φ is a continuous one-to-one strictly monotone and onto function with $\varphi(0) = 0$. Let $\alpha, \beta \in \mathbb{R}$. Then

$$\alpha < \beta \Leftrightarrow \alpha[-]\beta < 0, \tag{12}$$
$$\alpha \leqq \beta \Leftrightarrow \alpha[-]\beta \leqq 0, \tag{13}$$
$$\alpha[+]\beta < 0 \Rightarrow \alpha < (-1)[\cdot]\beta, \tag{14}$$
$$\alpha[+]\beta \leqq 0 \Rightarrow \alpha \leqq (-1)[\cdot]\beta. \tag{15}$$

Proof: We only prove (12) and (14), because (13) and (15) can be proven in the similar ways as that of (12) and (14), respectively. Without loss of generality, we assume that φ is strictly monotone increasing on $\mathbb{R}$.

The proof of (12). By the given conditions, we can see that

$$\begin{aligned}\alpha[-]\beta < 0 &\Leftrightarrow \varphi^{-1}(\varphi(\alpha) - \varphi(\beta)) < \varphi^{-1}(0) \\ &\Leftrightarrow \varphi(\alpha) - \varphi(\beta) < 0 \\ &\Leftrightarrow \varphi(\alpha) < \varphi(\beta) \\ &\Leftrightarrow \alpha < \beta.\end{aligned}$$

The proof of (14). It follows from (3) and (4) that

$$\begin{aligned}\alpha[+]\beta < 0 &\Rightarrow \varphi^{-1}(\varphi(\alpha) + \varphi(\beta)) < \varphi^{-1}(0) \\ &\Rightarrow \varphi(\alpha) + \varphi(\beta) < 0 \\ &\Rightarrow \varphi(\alpha) < -\varphi(\beta) \\ &\Rightarrow \alpha < \varphi^{-1}(-\varphi(\beta)) = (-1)[\cdot]\beta.\end{aligned}$$

Throughout of the rest of this paper, we further assume that h is a continuous one-to-one and onto function with h(0) = 0. Similarly, suppose that φ is a continuous one-to-one strictly monotone and onto function with $\varphi(0)=0$. Under the above assumptions, it is clear that $0[\cdot]\alpha=\alpha[\cdot]0=0$ *for any* $\alpha\in\mathbb{R}$.

Let X be a nonempty subset of $\mathbb{R}^n$. $F: X\times X\times\mathbb{R}^n\to\mathbb{R}$ is $(h;\varphi)$-sublinear and the functions $f=(f_1,\cdots,f_m): X\to\mathbb{R}^m$ and $g=(g_1,\cdots,g_p): X\to\mathbb{R}^p$ X are $(h;\varphi)$-differentiable on the set X, with respect to the same $(h;\varphi)$. Let $\rho=(\rho^1,\rho^2)$, where $\rho^1=(\rho_1^1,\cdots,\rho_m^1)\in\mathbb{R}^m$, $\rho^2=(\rho_1^2,\cdots,\rho_p^2)\in\mathbb{R}^p$. Let $\alpha=(\alpha^1,\alpha^2)$, where $\alpha^1: X\times X\to\mathbb{R}_+\backslash\{0\}$, $\alpha^2: X\times X\to\mathbb{R}_+\backslash\{0\}$, and let $d(\cdot,\cdot): X\times X\to\mathbb{R}$. Consider the following multi-objective programming problem:

$$(\text{MOP})_{h,\varphi}\ \min\ f(x)=(f_1(x),f_2(x),\cdots,f_m(x)),\ x\in X\subset\mathbb{R}^n$$
$$\text{s.t. } g(x)\leqq 0.$$

Let F denote the feasible solutions of $(\text{MOP})_{h,\varphi}$, assumed to be nonempty, that is,

$$F=\{x\in X:\quad g(x)\leqq 0\}.$$

Denote

$$M=\{1,2,\cdots,m\}\quad\text{and}\quad P=\{1,2,\cdots,p\}.$$

For a feasible solution $\bar{x}$, we denote by $J(\bar{x})$ the set

$$J(\bar{x})=\{j\in P:\quad g_j(\bar{x})=0\}.$$

In this paper, we consider the following weak Pareto solution of problem $(\text{MOP})_{h,\varphi}$. Weak Pareto solutions are often useful, since they are completely characterized by scalarization [22].

Definition 2.3 A point $\bar{x}$ is said to be a weak Pareto solution or weak minimum for $(\text{MOP})_{h,\varphi}$ if $\bar{x}\in F$ and $f(x)\not< f(\bar{x})$ for all $x\in F$.

Now, let us extend the notions of type I [9] and generalized type I [12,20-21] functions for $(\text{MOP})_{h,\varphi}$ using (F,α,ρ,d)-convexity presented in [13] in the setting of Ben-Tal's generalized algebraic operations.

Definition 2.4 For $i\in M$, (f_i,g) is said to be (h,φ)-(F,α,ρ,d)-type I at $\bar{x}\in X$, if for all $x\in F$ such that

$$f_i(x)[-]f_i(\bar{x})\geqq F(x,\bar{x};\alpha^1(x,\bar{x})\otimes\nabla^* f_i(\bar{x}))[+](\rho_i^1[\cdot]d^2(x,\bar{x})),\ i\in M, \tag{16}$$

and

$$(-1)[\cdot]g_j(\bar{x})\geqq F(x,\bar{x};\alpha^2(x,\bar{x})\otimes\nabla^* g_j(\bar{x}))[+](\rho_j^2[\cdot]d^2(x,\bar{x})),\ j\in P. \tag{17}$$

If in the above definition $x\neq\bar{x}$ and (16) is a strict inequality, then we say that (f_i,g) is semistrictly (h,φ)-(F,α,ρ,d)-type I at $\bar{x}$.

Definition 2.5 For $i \in M$, (f_i, g) is said to be quasi (h, φ)-(F, α, ρ, d)-type I at $\bar{x} \in X$, if for all $x \in F$ such that

$$[\sum_{i=1}^{m}] f_i(x) \leqq [\sum_{i=1}^{m}] f_i(\bar{x}) \Rightarrow$$

$$[\sum_{i=1}^{m}] F(x, \bar{x}; \alpha^1(x, \bar{x}) \otimes \nabla^* f_i(\bar{x}))[+][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(x, \bar{x})) \leqq 0,$$

and

$$(-1)[\cdot][\sum_{j=1}^{p}] g_j(\bar{x}) \leqq 0 \Rightarrow$$

$$[\sum_{j=1}^{p}] F(x, \bar{x}; \alpha^2(x, \bar{x}) \otimes \nabla^* g_j(\bar{x}))[+][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(x, \bar{x})) \leqq 0.$$

Definition 2.6 For $i \in M$, (f_i, g) is said to be pseudo (h, φ)-(F, α, ρ, d)-type I at $\bar{x} \in X$, if for all $x \in F$ such that

$$[\sum_{i=1}^{m}] F(x, \bar{x}; \alpha^1(x, \bar{x}) \otimes \nabla^* f_i(\bar{x}))[+][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(x, \bar{x})) \geqq 0 \Rightarrow$$

$$[\sum_{i=1}^{m}] f_i(x) \geqq [\sum_{i=1}^{m}] f_i(\bar{x}),$$

and

$$[\sum_{j=1}^{p}] F(x, \bar{x}; \alpha^2(x, \bar{x}) \otimes \nabla^* g_j(\bar{x}))[+][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(x, \bar{x})) \geqq 0 \Rightarrow$$

$$(-1)[\cdot][\sum_{j=1}^{p}] g_j(\bar{x}) \geqq 0.$$

Definition 2.7 For $i \in M$, (f_i, g) is said to be quasi pseudo (h, φ)-(F, α, ρ, d)-type I at $\bar{x} \in X$, if for all $x \in F$ such that

$$[\sum_{i=1}^{m}] f_i(x) \leqq [\sum_{i=1}^{m}] f_i(\bar{x}) \Rightarrow$$

$$[\sum_{i=1}^{m}] F(x, \bar{x}; \alpha^1(x, \bar{x}) \otimes \nabla^* f_i(\bar{x}))[+][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(x, \bar{x})) \leqq 0,$$

and

$$[\sum_{j=1}^{p}] F(x, \bar{x}; \alpha^2(x, \bar{x}) \otimes \nabla^* g_j(\bar{x}))[+][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(x, \bar{x})) \geqq 0 \Rightarrow$$

$$(-1)[\cdot][\sum_{j=1}^{p}] g_j(\bar{x}) \geqq 0.$$

If in the above definition $x \neq \bar{x}$ and (2.19) is satisfied as

$$[\sum_{j=1}^{p}]F(x,\bar{x};\alpha^2(x,\bar{x})\otimes\nabla^* g_j(\bar{x}))[+][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(x,\bar{x}))\geqq 0 \Rightarrow$$

$$(-1)[\cdot][\sum_{j=1}^{p}]g_j(\bar{x})>0,$$

then we say that (f_i, g) is quasi strictly pseudo (h,φ)-(F,α,ρ,d)-type I at $\bar{x}$.

Definition 2.8 For $i\in M$, (f_i, g) is said to be pseudo quasi (h,φ)-(F,α,ρ,d)-type I at $\bar{x}\in X$, if for all $x\in F$ such that

$$[\sum_{i=1}^{m}]F(x,\bar{x};\alpha^1(x,\bar{x})\otimes\nabla^* f_i(\bar{x}))[+][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(x,\bar{x}))\geqq 0 \Rightarrow$$

$$[\sum_{i=1}^{m}]f_i(x)\geqq[\sum_{i=1}^{m}]f_i(\bar{x}),$$

and

$$(-1)[\cdot][\sum_{j=1}^{p}]g_j(\bar{x})\leqq 0 \Rightarrow$$

$$[\sum_{j=1}^{p}]F(x,\bar{x};\alpha^2(x,\bar{x})\otimes\nabla^* g_j(\bar{x}))[+][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(x,\bar{x}))\leqq 0.$$

If in the above definition $x \neq \bar{x}$ and (2.21) is satisfied as

$$[\sum_{i=1}^{m}]F(x,\bar{x};\alpha^1(x,\bar{x})\otimes\nabla^* f_i(\bar{x}))[+][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(x,\bar{x}))\geqq 0 \Rightarrow$$

$$[\sum_{i=1}^{m}]f_i(x)>[\sum_{i=1}^{m}]f_i(\bar{x}),$$

then we say that (f_i, g) is strictly pseudo quasi (h,φ)-(F,α,ρ,d)-type I at $\bar{x}$.

3 Sufficient Optimality Conditions

In this section, we establish sufficient optimality conditions for a feasible solution $\bar{x}$ to be a weak minimum for $(\text{MOP})_{h,\varphi}$ under the (h,φ)-(F,α,ρ,d)-type I and pseudo quasi (h,φ)-(F,α,ρ,d)-type I assumptions.

Theorem 3.1 Suppose that there exist a feasible solution $\bar{x}\in F$ and $\bar{\lambda}=(\bar{\lambda}_1,\cdots,\bar{\lambda}_m)\in\mathbb{R}^m$, $\bar{\lambda}\geq 0$, $\bar{\mu}_j\geqq 0$, $j\in J(\bar{x})$ such that

$$\left(\overset{m}{\underset{i=1}{\oplus}}(\bar{\lambda}_i\otimes\nabla^* f_i(\bar{x}))\right)\oplus\left(\underset{j\in J(x)}{\oplus}(\bar{\mu}_j\otimes\nabla^* g_j(\bar{x}))\right)=0. \tag{18}$$

If for $i \in M, \left(f_i, g_{J(x)}\right)$ is (h, φ)-(F, α, ρ, d)-type I at $\bar{x}$ with

$$\left(\alpha^1(\cdot,\bar{x})^{-1}[\cdot][\sum_{i=1}^{m}]((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\cdot,\bar{x}))\right)[+]\left(\alpha^2(\cdot,\bar{x})^{-1}[\cdot][\sum_{j\in J(\bar{x})}]((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\cdot,\bar{x}))\right) \geqq 0, \tag{19}$$

where $g_{J(\bar{x})} = (g_j)_{j\in J(\bar{x})}$. Then $\bar{x}$ is a weak minimum for $(\text{MOP})_{h,\varphi}$.

Proof: Since (18) holds, by the (h, φ)-sublinearity of F, for any $x \in X$ we get

$$F\left(x, \bar{x}; \left(\overset{m}{\underset{i=1}{\oplus}}(\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x}))\right) \oplus \left(\underset{j\in J(\bar{x})}{\oplus}(\bar{\mu}_j \otimes \nabla^* g_j(\bar{x}))\right)\right) = 0. \tag{20}$$

We proceed by contradiction. Suppose that $\bar{x}$ is not a weak minimum of $(\text{MOP})_{h,\varphi}$, then there is a feasible solution $\hat{x}$ of $(\text{MOP})_{h,\varphi}$ such that

$$f_i(\hat{x}) < f_i(\bar{x}) \quad i \in M.$$

Since $\bar{\lambda} \geq 0$, then there is at least an index k such that

$$\bar{\lambda}_k > 0, \quad \bar{\lambda}_i \geqq 0 \quad \text{for all } i \in M, i \neq k.$$

From Lemma 2.3 (a)-(c), we get

$$\bar{\lambda}_k[\cdot]f_k(\hat{x}) < \bar{\lambda}_i[\cdot]f_i(\bar{x}),$$
$$\bar{\lambda}_i[\cdot]f_i(\hat{x}) \leqq \bar{\lambda}_i[\cdot]f_i(\bar{x}) \quad \text{for all } i \in M, i \neq k.$$

It follows from Lemma 2.3 (e) and (12) that

$$[\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\hat{x})) < [\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\bar{x})), \ [\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\hat{x}))[-][\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\bar{x})) < 0. \tag{21}$$

By (h, φ)-(F, α, ρ, d)-type I assumption on $(f_i, g_{J(\bar{x})})$, for above $\hat{x}$ we have

$$f_i(\hat{x})[-]f_i(\bar{x}) \geqq F(\hat{x}, \bar{x}; \alpha^1(\hat{x}, \bar{x}) \otimes \nabla^* f_i(\bar{x}))[+](\rho_i^1[\cdot]d^2(\hat{x}, \bar{x})), \quad i \in M,$$

and

$$0 = (-)[\cdot]g_j(\bar{x}) \geqq F(\hat{x}, \bar{x}; \alpha^2(\hat{x}, \bar{x}) \otimes \nabla^* g_j(\bar{x}))[+](\rho_j^2[\cdot]d^2(\hat{x}, \bar{x})), \quad j \in J(\bar{x}).$$

From the (h, φ)-sublinearity of F, we further get

$$f_i(\hat{x})[-]f_i(\bar{x}) \geqq \alpha^1(\hat{x}, \bar{x})[\cdot]\, F(\hat{x}, \bar{x}; \nabla^* f_i(\bar{x}))[+](\rho_i^1[\cdot]d^2(\hat{x}, \bar{x})), \quad i \in M,$$

and

$$0 \geqq \alpha^2(\hat{x}, \bar{x})[\cdot]\, F(\hat{x}, \bar{x}; \nabla^* g_j(\bar{x}))[+](\rho_j^2[\cdot]d^2(\hat{x}, \bar{x})), \quad j \in J(\bar{x}).$$

Since $\bar{\lambda}_i \geqq 0, i \in M$ $\bar{\mu}_j \geqq 0$, $j \in J(\bar{x})$, by Lemma 2.3 (a), we have

$$\bar{\lambda}_i[\cdot]\left(f_i(\hat{x})[-]f_i(\bar{x})\right) \geqq \bar{\lambda}_i[\cdot]\left(\alpha^1(\hat{x}, \bar{x})[\cdot]F(\hat{x}, \bar{x}; \nabla^* f_i(\bar{x}))[+](\rho_i^1[\cdot]d^2(\hat{x}, \bar{x}))\right), \quad i \in M,$$

and

$$0 \geqq \bar{\mu}_j[\cdot]\left(\alpha^2(\hat{x},\bar{x})[\cdot]\, F(\hat{x},\bar{x};\nabla^* g_j(\bar{x}))[+](\rho_j^2[\cdot]d^2(\hat{x},\bar{x}))\right),\ \ j \in J(\bar{x}).$$

From Lemma 2.2 (a) − (c), we get

$$\left(\bar{\lambda}_i[\cdot]f_i(\hat{x})\right)[-]\left(\bar{\lambda}_i[\cdot]f_i(\bar{x})\right) \geqq \alpha^1(\hat{x},\bar{x})[\cdot]\left(\bar{\lambda}_i[\cdot]F(\hat{x},\bar{x};\nabla^* f_i(\bar{x}))\right)[+]\left((\bar{\lambda}_i\rho_i^1[\cdot]d^2(\hat{x},\bar{x})\right),\ \ i \in M,$$

and

$$0 \geqq \alpha^2(\hat{x},\bar{x})[\cdot]\left(\ \bar{\mu}_j[\cdot]F(\hat{x},\bar{x};\nabla^* g_j(\bar{x}))\right)[+]\left((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\hat{x},\bar{x})\right),\ \ j \in J(\bar{x}).$$

By the (h, φ)-sublinearity of F again, we have

$$\left(\bar{\lambda}_i[\cdot]f_i(\hat{x})\right)[-]\left(\bar{\lambda}_i[\cdot]f_i(\bar{x})\right) \geqq \alpha^1(\hat{x},\bar{x})[\cdot]F(\hat{x},\bar{x};\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x}))[+]\left((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\hat{x},\bar{x})\right),\ \ i \in M,$$

and

$$0 \geqq \alpha^2(\hat{x},\bar{x})[\cdot]F\left(\hat{x},\bar{x};\bar{\mu}_j \otimes \nabla^* g_j(\bar{x})\right)[+]\left((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\hat{x},\bar{x})\right),\ \ j \in J(\bar{x}).$$

By Lemma 2.3 (e), we get

$$[\sum_{i=1}^{m}]\left((\bar{\lambda}_i[\cdot]f_i(\hat{x}))[-](\bar{\lambda}_i[\cdot]f_i(\bar{x}))\right) \geqq [\sum_{i=1}^{m}]\left((\alpha^1(\hat{x},\bar{x})[\cdot]F(\hat{x},\bar{x};\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x}))[+]((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\hat{x},\bar{x}))\right),$$

and

$$0 \geqq [\sum_{j\in J(\bar{x})}]\left(\alpha^2(\hat{x},\bar{x})[\cdot]F(\hat{x},\bar{x};\bar{\mu}_j \otimes \nabla^* g_j(\bar{x}))[+]((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\hat{x},\bar{x}))\right).$$

Furthermore, we get from Lemma 2.2 (d) that

$$\begin{aligned} &[\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\hat{x}))[-][\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\bar{x})) \geqq \\ &[\sum_{i=1}^{m}]\left(\alpha^1(\hat{x},\bar{x})[\cdot]F(\hat{x},\bar{x};\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x}))\right)[+][\sum_{i=1}^{m}]\left((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\hat{x},\bar{x})\right), \end{aligned} \tag{22}$$

and

$$0 \geqq [\sum_{j\in J(\bar{x})}]\left(\alpha^2(\hat{x},\bar{x})[\cdot]F(\hat{x},\bar{x};\bar{\mu}_j \otimes \nabla^* g_j(\bar{x}))\right)[+][\sum_{j\in J(\bar{x})}]\left((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\hat{x},\bar{x})\right). \tag{23}$$

It yields from (21) and (22) that

$$[\sum_{i=1}^{m}]\left(\alpha^1(\hat{x},\bar{x})[\cdot]F(\hat{x},\bar{x};\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x}))\right)[+][\sum_{i=1}^{m}]\left((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\hat{x},\bar{x})\right) < 0. \tag{24}$$

By (24) and (23), it follows from (14) and (15) that

$$[\sum_{i=1}^{m}]\left(\alpha^1(\hat{x},\bar{x})[\cdot]F(\hat{x},\bar{x};\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x}))\right) < (-1)[\cdot][\sum_{i=1}^{m}]\left((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\hat{x},\bar{x})\right),$$

$$[\sum_{j\in J(\bar{x})}]\left(\alpha^2(\hat{x},\bar{x})[\cdot]F(\hat{x},\bar{x};\bar{\mu}_j\otimes\nabla^* g_j(\bar{x}))\right)\leqq(-1)[\cdot][\sum_{j\in J(\bar{x})}]\left((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\hat{x},\bar{x})\right).$$

We further get from Lemma 2.2 (b) that

$$\alpha^1(\hat{x},\bar{x})[\cdot][\sum_{i=1}^{m}]F(\hat{x},\bar{x};\bar{\lambda}_i\otimes\nabla^* f_i(\bar{x}))<(-1)[\cdot][\sum_{i=1}^{m}]\left((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\hat{x},\bar{x})\right),\tag{25}$$

$$\alpha^2(\hat{x},\bar{x})[\cdot][\sum_{j\in J(\bar{x})}]F(\hat{x},\bar{x};\bar{\mu}_j\otimes\nabla^* g_j(\bar{x}))\leqq(-1)[\cdot][\sum_{j\in J(\bar{x})}]\left((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\hat{x},\bar{x})\right).\tag{26}$$

Multiplying (25) and (26) with $\alpha^1(\hat{x},\bar{x})^{-1}$ and $\alpha^2(\hat{x},\bar{x})^{-1}$ in sense of φ-scalar multiplication, respectively, we get from Lemma 2.2 (a) and Lemma 2.3 (a) and (c)that

$$\begin{aligned}&\left(\alpha^1(\hat{x},\bar{x})^{-1}\alpha^1(\hat{x},\bar{x})\right)[\cdot][\sum_{i=1}^{m}]F(\hat{x},\bar{x};\bar{\lambda}_i\otimes\nabla^* f_i(\bar{x}))<\\&(-1)[\cdot]\left(\alpha^1(\hat{x},\bar{x})^{-1}[\cdot][\sum_{i=1}^{m}]((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\hat{x},\bar{x}))\right),\end{aligned}\tag{27}$$

$$\begin{aligned}&\left(\alpha^2(\hat{x},\bar{x})^{-1}\alpha^2(\hat{x},\bar{x})\right)[\cdot][\sum_{j\in J(\bar{x})}]F(\hat{x},\bar{x};\bar{\mu}_j\otimes\nabla^* g_j(\bar{x}))\leqq\\&(-1)[\cdot]\left(\alpha^2(\hat{x},\bar{x})^{-1}[\cdot][\sum_{j\in J(\bar{x})}]((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\hat{x},\bar{x}))\right).\end{aligned}\tag{28}$$

Consequently, we get from Lemma 2.2 (b) and Lemma 2.3 (e) that

$$\begin{aligned}&[\sum_{i=1}^{m}]F(\hat{x},\bar{x};\bar{\lambda}_i\otimes\nabla^* f_i(\bar{x}))[+][\sum_{j\in J(\bar{x})}]F(\hat{x},\bar{x};\bar{\mu}_j\otimes\nabla^* g_j(\bar{x}))<(-1)[\cdot]\Big(\big(\alpha^1(\hat{x},\bar{x})^{-1}\\&[\cdot][\sum_{i=1}^{m}]((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\hat{x},\bar{x}))\big)[+]\big(\alpha^2(\hat{x},\bar{x})^{-1}[\cdot][\sum_{j\in J(\bar{x})}]((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\hat{x},\bar{x}))\big)\Big)\end{aligned}\tag{29}$$

By (19), we get from Lemma 2.3 (d) that

$$\begin{aligned}&(-1)[\cdot]\left(\left(\alpha^1(\hat{x},\bar{x})^{-1}[\cdot][\sum_{i=1}^{m}]((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\hat{x},\bar{x}))\right)[+]\right.\\&\left.\left(\alpha^2(\hat{x},\bar{x})^{-1}[\cdot][\sum_{j\in J(\bar{x})}]((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\hat{x},\bar{x}))\right)\right)\leqq 0.\end{aligned}\tag{30}$$

We get from (29) and (30) that

$$[\sum_{i=1}^{m}]F(\hat{x},\bar{x};\bar{\lambda}_i\otimes\nabla^* f_i(\bar{x}))[+][\sum_{j\in J(\bar{x})}]F(\hat{x},\bar{x};\bar{\mu}_j\otimes\nabla^* g_j(\bar{x}))<0$$

By the (h, φ)-sublinearity of F and (9), noticing that (20) holds, we summarize to get

$$0 = F\left(\hat{x}, \overline{x}; \left(\overset{m}{\underset{i=1}{\oplus}}(\overline{\lambda}_i \otimes \nabla^* f_i(\overline{x}))\right) \oplus \left(\underset{j\in J(\overline{x})}{\oplus}(\overline{\mu}_j \otimes \nabla^* g_j(\overline{x}))\right)\right)$$

$$\leqq [\sum_{i=1}^{m}] F(\hat{x}, \overline{x}; \overline{\lambda}_i \otimes \nabla^* f_i(\overline{x}))[+][\sum_{j\in J(\overline{x})}] F(\hat{x}, \overline{x}; \overline{\mu}_j \otimes \nabla^* g_j(\overline{x})) < 0,$$

we obtain a contradiction. Hence, $\overline{x}$ is a weak minimum for $(MOP)_{h,\varphi}$.

Theorem 3.2 Suppose that there exist $\overline{x} \in F$ and $\overline{\lambda} = (\overline{\lambda}_1, \cdots, \overline{\lambda}_m) \in \mathbb{R}^m$, $\overline{\lambda} \geq 0$, $\overline{\mu}_j \geqq 0$, $j \in J(\overline{x})$ such that (18) is satisfied. If for $i \in M$, $(\overline{\lambda}_i[\cdot]f_i, \overline{G}_{J(\overline{x})})$ is pseudo quasi (h, φ)-(F, α, ρ, d)-type I at $\overline{x}$ with

$$\left(\alpha^1(\cdot, \overline{x})^{-1}[\cdot][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(\cdot, \overline{x}))\right)[+]\left(\alpha^2(\cdot, \overline{x})^{-1}[\cdot][\sum_{j\in J(\overline{x})}](\rho_j^2[\cdot]d^2(\cdot, \overline{x}))\right) \geqq 0, \tag{31}$$

where $\overline{G}_{J(\overline{x})} = (\overline{G}_j)_{j\in J(\overline{x})}$, $\overline{G}_j = \overline{\mu}_j[\cdot]g_j$, then $\overline{x}$ is a weak minimum for $(MOP)_{h,\varphi}$.

Proof: Suppose that $\overline{x}$ is not a weak minimum for $(MOP)_{h,\varphi}$, then there exists a feasible solution $\overline{x}$ for $(MOP)_{h,\varphi}$ such that

$$f_i(\hat{x}) < f_i(\overline{x}) \ \ i \in M.$$

Noticing that $\overline{\lambda} \geq 0$, by the same argument as in that of Theorem 3.1, we can get

$$[\sum_{i=1}^{m}](\overline{\lambda}_i[\cdot]f_i(\hat{x})) < [\sum_{i=1}^{m}](\overline{\lambda}_i[\cdot]f_i(\overline{x})). \tag{32}$$

On the other hand, Since $g_j(\overline{x}) = 0$, $\overline{\mu}_j \geqq 0$, $\overline{\mu}_j \geqq 0$, $j \in J(\overline{x})$, we get from Lemma 2.3 (a), (d) and (e) that

$$(-1)[\cdot][\sum_{j\in J(\overline{x})}]\overline{\mu}_j[\cdot]g_j(\overline{x}) \leqq 0. \tag{33}$$

By the pseudo quasi (h, φ)-(F, α, ρ, d)-type I hypothesis on $(\overline{\lambda}_i[\cdot]f_i, \overline{G}_{J(\overline{x})})$ for $i \in M$ at $\overline{x}$, for above $\hat{x}$, we get from (32) and (33) that

$$[\sum_{i=1}^{m}] F\left(\hat{x}, \overline{x}; \alpha^1(\hat{x}, \overline{x}) \otimes \nabla^*(\overline{\lambda}_i[\cdot]f_i(\overline{x}))\right)[+][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(\hat{x}, \overline{x})) < 0,$$

and

$$[\sum_{j\in J(\overline{x})}] F\left(\hat{x}, \overline{x}; \alpha^2(\hat{x}, \overline{x}) \otimes \nabla^*(\overline{\mu}_j[\cdot]g_j(\overline{x}))\right)[+][\sum_{j\in J(\overline{x})}](\rho_j^2[\cdot]d^2(\hat{x}, \overline{x})) \leqq 0.$$

By the (h, φ)-sublinearity of F, we get from Lemma 2.1 (c) and Lemma 2.2 (b) that

$$\alpha^1(\hat{x}, \overline{x})[\cdot][\sum_{i=1}^{m}] F\left(\hat{x}, \overline{x}; \overline{\lambda}_i \otimes \nabla^* f_i(\overline{x})\right)[+][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(\hat{x}, \overline{x})) < 0,$$

and

$$\alpha^2(\hat{x},\overline{x})[\cdot][\sum_{j\in J(x)}]F\left(\hat{x},\overline{x};\overline{\mu}_j\otimes\nabla^* g_j(\overline{x})\right)[+][\sum_{j\in J(\overline{x})}](\rho_j^2[\cdot]d^2(\hat{x},\overline{x}))\leqq 0.$$

It yields from (14) and (15) that

$$\alpha^1(\hat{x},\overline{x})[\cdot][\sum_{i=1}^{m}]F\left(\hat{x},\overline{x};\overline{\lambda}_i\otimes\nabla^* f_i(\overline{x})\right)<(-1)[\cdot][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(\hat{x},\overline{x})), \tag{34}$$

and

$$\alpha^2(\hat{x},\overline{x})[\cdot][\sum_{j\in J(\overline{x})}]F\left(\hat{x},\overline{x};\overline{\mu}_j\otimes\nabla^* g_j(\overline{x})\right)\leqq(-1)[\cdot][\sum_{j\in J(\overline{x})}](\rho_j^2[\cdot]d^2(\hat{x},\overline{x})). \tag{35}$$

Multiplying (34) and (35) with $\alpha^1(\hat{x},\overline{x})^{-1}$ and $\alpha^2(\hat{x},\overline{x})^{-1}$ in sense of φ-scalar multiplication, respectively, we get from Lemma 2.2 (a) and Lemma 2.3 (a) and (c) that

$$[\sum_{i=1}^{m}]F\left(\hat{x},\overline{x};\overline{\lambda}_i\otimes\nabla^* f_i(\overline{x})\right)<(-1)[\cdot]\left(\alpha^1(\hat{x},\overline{x})^{-1}[\cdot][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(\hat{x},\overline{x}))\right),$$

and

$$[\sum_{j\in J(x)}]F\left(\hat{x},\overline{x};\overline{\mu}_j\otimes\nabla^* g_j(\overline{x})\right)\leqq(-1)[\cdot]\left(\alpha^2(\hat{x},\overline{x})[\cdot][\sum_{j\in J(x)}](\rho_j^2[\cdot]d^2(\hat{x},\overline{x}))\right).$$

Hence, by Lemma 2.3 (e) and Lemma 2.2 (b), the above two inequalities give that

$$[\sum_{i=1}^{m}]F\left(\hat{x},\overline{x};\overline{\lambda}_i\otimes\nabla^* f_i(\overline{x})\right)[+][\sum_{j\in J(\overline{x})}]F\left(\hat{x},\overline{x};\overline{\mu}_j\otimes\nabla^* g_j(\overline{x})\right)<(-1)[\cdot]$$
$$\left(\left(\alpha^1(\hat{x},\overline{x})^{-1}[\cdot][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(\hat{x},\overline{x}))\right)[+]\left(\alpha^2(\hat{x},\overline{x})^{-1}[\cdot][\sum_{j\in J(\overline{x})}](\rho_j^2[\cdot]d^2(\hat{x},\overline{x}))\right)\right).$$

By (31), we get from Lemma 2.3 (d) that

$$(-1)[\cdot]\left(\left(\alpha^1(\hat{x},\overline{x})^{-1}[\cdot][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(\hat{x},\overline{x}))\right)[+]\right.$$
$$\left.\left(\alpha^2(\hat{x},\overline{x})^{-1}[\cdot][\sum_{j\in J(\overline{x})}](\rho_j^2[\cdot]d^2(\hat{x},\overline{x}))\right)\right)\leqq 0.$$

Furthermore, the above two inequalities give that

$$[\sum_{i=1}^{m}]F\left(\hat{x},\overline{x};\overline{\lambda}_i\otimes\nabla^* f_i(\overline{x})\right)[+][\sum_{j\in J(\overline{x})}]F\left(\hat{x},\overline{x};\overline{\mu}_j\otimes\nabla^* g_j(\overline{x})\right)<0.$$

By the (h,φ)-sublinearity of F and (9), noticing that (18) holds, we summarize to get

$$0 = F\left(\hat{x}, \bar{x}; \left(\overset{m}{\underset{i=1}{\oplus}} (\bar{\lambda}_i \otimes \nabla^* f_i(\bar{x}))\right) \oplus \left(\underset{j \in J(\bar{x})}{\oplus} (\bar{\mu}_j \otimes \nabla^* g_j(\bar{x}))\right)\right)$$

$$\leqq [\sum_{i=1}^{m}] F(\hat{x}, \bar{x}; \bar{\lambda}_i \otimes \nabla^* f_i(\bar{x}))[+][\sum_{j \in J(\bar{x})}] F(\hat{x}, \bar{x}; \bar{\mu}_j \otimes \nabla^* g_j(\bar{x})) < 0,$$

which is a contradiction. This completes the proof.

4 Duality Results

Now, in relation to $(MOP)_{h,\varphi}$ we consider the following Mond and Weir type dual under the (h, φ)-(F, α, ρ, d)-type I and generalized (h, φ)-(F, α, ρ, d)-type I assumptions.

$$(\text{DMOP})_{h,\varphi} \max\ f(y) = (f_1(y), f_2(y), \cdots, f_m(y))$$

$$\text{s.t.} \left(\overset{m}{\underset{i=1}{\oplus}} (\bar{\lambda}_i \otimes \nabla^* f_i(y))\right) \oplus \left(\sum_{j=1}^{p} (\bar{\mu}_j \otimes \nabla^* g_j(y))\right) = 0, \tag{36}$$

$$[\sum_{j=1}^{p}](\bar{\mu}_j [\cdot] g_j(y)) \geqq 0, \tag{37}$$

$$\lambda = (\lambda_1, \cdots, \lambda_m),\ \ \lambda \geqq 0, \tag{38}$$

$$\mu = (\mu_1, \cdots, \mu_p),\ \ \mu \geqq 0. \tag{39}$$

We shall provide the weak and converse duality relations between problems $(MOP)_{h,\varphi}$ and $(\text{DMOP})_{h,\varphi}$.

Theorem 4.1 (Weak Duality) Let x and (y, λ, μ) be feasible solutions for $(MOP)_{h,\varphi}$ and $(\text{DMOP})_{h,\varphi}$, respectively. If any of the following holds:

(I) For $i \in M$, (f_i, g) is (h, φ)-(F, α, ρ, d)-type I at y with

$$\begin{gathered} \left(\alpha^1(\cdot, y)^{-1}[\cdot][\sum_{i=1}^{m}]((\lambda_i \rho_i^1)[\cdot] d^2(\cdot, y))\right)[+] \\ \left(\alpha^2(\cdot, y)^{-1}[\cdot][\sum_{j=1}^{p}]((\mu_i \rho_j^2)[\cdot] d^2(\cdot, y))\right) \geqq 0, \end{gathered} \tag{40}$$

(II) For $i \in M$, $(\lambda_i[\cdot] f_i, G)$ is pseudo quasi (h, φ)-(F, α, ρ, d)-type I at y with

$$\left(\alpha^1(\cdot, y)^{-1}[\cdot][\sum_{i=1}^{m}](\rho_i^1[\cdot] d^2(\cdot, y))\right)[+]\left(\alpha^2(\cdot, y)^{-1}[\cdot][\sum_{j=1}^{p}]\rho_j^2[\cdot] d^2(\cdot, y))\right) \geqq 0, \tag{41}$$

where $G = (G_1, G_2, \cdots, G_p)$, $G_j \geqq \mu_j[\cdot] g_j$, $j \in P$.

Then

$$f(x) \not< f(y).$$

Proof: We proceed by contradiction. Suppose that

$$f(x) < f(y).$$

Since $\lambda \geq 0$, by the similar argument as in that of Theorem 3.1, we can get

$$[\sum_{i=1}^{m}](\lambda_i[\cdot]f_i(x)) < [\sum_{i=1}^{m}](\lambda_i[\cdot]f_i(y)), \tag{42}$$

$$[\sum_{i=1}^{m}](\lambda_i[\cdot]f_i(x))[-][\sum_{i=1}^{m}](\lambda_i[\cdot]f_i(y)) < 0. \tag{43}$$

On the other hand, we have from Lemma 2.3 (d) and (37) that

$$(-1)[\cdot][\sum_{j=1}^{p}](\mu_j[\cdot]g_i(y)) \leq 0. \tag{44}$$

Assuming that condition (I) holds, we get from Definition 2.4 that

$$f_i(x)[-]f_i(y) \geqq F\left(x, y; \alpha^1(x, y) \otimes \nabla^* f_i(y)\right)[+](\rho_i^1[\cdot]d^2(x, y)), \; i \in M,$$

and

$$(-1)[\cdot]g_j(y) \geqq F\left(x, y; \alpha^2(x, y) \otimes \nabla^* g_j(y)\right)[+](\rho_j^2[\cdot]d^2(x, y)), \; j \in P.$$

With the similar argument as that of Theorem 3.1 we can get

$$[\sum_{i=1}^{m}](\lambda_i[\cdot]f_i(x))[-][\sum_{i=1}^{m}](\lambda_i[\cdot]f_i(y)) \geqq$$

$$\alpha^1(x, y)[\cdot][\sum_{i=1}^{m}]F(x, y; \lambda_i \otimes \nabla^* f_i(y))[+][\sum_{i=1}^{m}]((\lambda_i\rho_i^1[\cdot]d^2(x, y)),$$

and

$$(-1)[\cdot][\sum_{j=1}^{p}](\mu_j[\cdot]g_j(y)) \geqq$$

$$\alpha^2(x, y)[\cdot][\sum_{j=1}^{p}]F(x, y; \mu_j \otimes \nabla^* g_j(y))[+][\sum_{j=1}^{p}]((\mu_j\rho_j^2[\cdot]d^2(x, y)).$$

From (43) and (44), the above two inequalities give that

$$\alpha^1(x, y)[\cdot][\sum_{i=1}^{m}]F(x, y; \lambda_i \otimes \nabla^* f_i(y))[+][\sum_{i=1}^{m}]((\lambda_i\rho_i^1[\cdot]d^2(x, y)) < 0,$$

and

$$\alpha^2(x, y)[\cdot][\sum_{j=1}^{p}]F(x, y; \mu_j \otimes \nabla^* g_j(y))[+][\sum_{j=1}^{p}]((\mu_j\rho_j^2[\cdot]d^2(x, y)) \leqq 0.$$

It follows from (14) and (15) that

$$\alpha^1(x, y)[\cdot][\sum_{i=1}^{m}]F(x, y; \lambda_i \otimes \nabla^* f_i(y)) < (-1)[\cdot][\sum_{i=1}^{m}]\big((\lambda_i\rho_i^1)[\cdot]d^2(x, y)\big), \tag{45}$$

$$\alpha^2(x, y)[\cdot][\sum_{j=1}^{p}]F(x, y; \mu_j \otimes \nabla^* g_j(y)) \leqq (-1)[\cdot][\sum_{j=1}^{p}]((\mu_j\rho_j^2)[\cdot]d^2(x, y)). \tag{46}$$

Multiplying (45) and (46) with $\alpha^1(x, y)^{-1}$ and $\alpha^2(x, y)^{-1}$ in sense of φ-scalar multiplication, respectively, we get from Lemma 2.2 (a) and Lemma 2.3 (a) and (c) that

$$[\sum_{i=1}^{m}]F(x, y; \lambda_i \otimes \nabla^* f_i(y)) < (-1)[\cdot]\left(\alpha^1(x, y)^{-1}[\cdot][\sum_{i=1}^{m}]((\lambda_i\rho_i^1)[\cdot]d^2(x, y))\right),$$

and

$$[\sum_{j=1}^{p}]F(x, y; \mu_j \otimes \nabla^* g_j(y)) \leqq (-1)[\cdot]\left(\alpha^2(x, y)^{-1}[\cdot][\sum_{j=1}^{p}]((\mu_j\rho_j^2)[\cdot]d^2(x, y))\right).$$

Hence, we get from Lemma 2.2 (b) and Lemma 2.3 (e) that

$$\begin{aligned}&[\sum_{i=1}^{m}]F(x, y; \lambda_i \otimes \nabla^* f_i(y))[+][\sum_{j=1}^{p}]F(x, y; \mu_j \otimes \nabla^* g_j(y)) < (-1)[\cdot]\\&\left(\left(\alpha^1(x, y)^{-1}[\cdot][\sum_{i=1}^{m}]((\lambda_i\rho_i^1)[\cdot]d^2(x, y))\right)[+]\left(\alpha^2(x, y)^{-1}[\cdot][\sum_{j=1}^{p}]((\mu_j\rho_j^2)[\cdot]d^2(x, y))\right)\right).\end{aligned} \tag{47}$$

By (40), we get from Lemma 2.3 (d) that

$$\begin{aligned}(-1)[\cdot]&\left(\left(\alpha^1(x, y)^{-1}[\cdot][\sum_{i=1}^{m}]((\lambda_i\rho_i^1)[\cdot]d^2(x, y))\right)[+]\right.\\&\left.\left(\alpha^2(x, y)^{-1}[\cdot][\sum_{j=1}^{p}]((\mu_j\rho_j^2)[\cdot]d^2(x, y))\right)\right) \leqq 0.\end{aligned} \tag{48}$$

Combing (47) with (48), we get

$$[\sum_{i=1}^{m}]F(x, y; \lambda_i \otimes \nabla^* f_i(y))[+][\sum_{j=1}^{p}]F(x, y; \mu_j \otimes \nabla^* g_j(y)) < 0.$$

By the (h, φ)-sublinearity of F and (9), noticing that (36) holds, we summarize to get

$$\begin{aligned}0 &= F\left(x, y; \left(\bigoplus_{i=1}^{m}(\lambda_i \otimes \nabla^* f_i(y))\right) \oplus \left(\bigoplus_{j=1}^{p}(\mu_j \otimes \nabla^* g_j(y))\right)\right)\\&\leqq [\sum_{i=1}^{m}]F(x, y; \lambda_i \otimes \nabla^* f_i(y))[+][\sum_{j=1}^{p}]F(x, y; \mu_j \otimes \nabla^* g_j(y)) < 0\end{aligned}$$

This is a contradiction.

By condition (II), we get from definition 2.8 that

$$[\sum_{i=1}^{m}]F(x, y; \alpha^1(x, y) \otimes \nabla^*(\lambda_i[\cdot]f_i)(y))[+][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(x, y)) \geqq 0 \Rightarrow$$
$$[\sum_{i=1}^{m}](\lambda_i[\cdot]f_i(x)) \geqq [\sum_{i=1}^{m}](\lambda_i[\cdot]f_i(y)), \tag{49}$$

and

$$(-1)[\cdot][\sum_{j=1}^{p}](\mu_j[\cdot]g_j(y)) \leqq 0 \Rightarrow$$
$$[\sum_{j=1}^{p}]F(x, y; \alpha^2(x, y) \otimes \nabla^*(\mu_j[\cdot]g_j)(y))[+][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(x, y)) \leqq 0. \tag{50}$$

By (h, φ)-sublinearity of F, Lemma 2.1 (c) and Lemma 2.2 (b), (49) and (50) give that

$$\alpha^1(x, y)[\cdot][\sum_{i=1}^{m}]F(x, y; \lambda_i \otimes \nabla^* f_i(y))[+][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(x, y)) \geqq 0 \Rightarrow$$
$$[\sum_{i=1}^{m}](\lambda_i[\cdot]f_i(x)) \geqq [\sum_{i=1}^{m}](\lambda_i[\cdot]f_i(y)), \tag{51}$$

and

$$(-1)[\cdot][\sum_{j=1}^{p}](\mu_j[\cdot]g_i(y)) \leqq 0 \Rightarrow$$
$$\alpha^2(x, y)[\cdot][\sum_{j=1}^{p}]F(x, y; \mu_j \otimes \nabla^* g_j(y))[+][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(x, y)) \leqq 0. \tag{52}$$

Making use of (42) and (44), we get from (51) and (52), respectively, that

$$\alpha^1(x, y)[\cdot][\sum_{i=1}^{m}]F(x, y; \lambda_i \otimes \nabla^* f_i(y))[+][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(x, y)) < 0$$

and

$$\alpha^2(x, y)[\cdot][\sum_{j=1}^{p}]F(x, y; \mu_j \otimes \nabla^* g_j(y))[+][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(x, y)) \leqq 0.$$

The rest of proofs follow the lines of Theorem 3.2.

Theorem 4.2 (Converse Duality). Let $\bar{x}$ be a weak minimum for $(\mathrm{MOP})_{h,\varphi}$ and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a weak maximum for $(\mathrm{DMOP})_{h,\varphi}$ such that

$$[\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\bar{x})) \leqq [\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\bar{y})) \tag{53}$$

If any of the following holds:

(I) For $i \in M$, (f_i, g) is semistrictly (h, φ)-(F, α, ρ, d)-type I at $\bar{y}$ with

$$\left(\alpha^1(\cdot, \bar{y})^{-1}[\cdot][\sum_{i=1}^{m}]((\bar{\lambda}_i \rho_i^1)[\cdot]d^2(\cdot, \bar{y}))\right)[+] \left(\alpha^2(\cdot, \bar{y})^{-1}[\cdot][\sum_{j=1}^{p}]((\bar{\mu}_j \rho_j^2)[\cdot]d^2(\cdot, \bar{y}))\right) \geqq 0, \tag{54}$$

(II) For $i \in M$, $(\bar{\lambda}_i[\cdot]f_i, \bar{G})$ is strictly pseudo quasi (h, φ)-(F, α, ρ, d)-type I at $\bar{y}$ with

$$\left(\alpha^1(\cdot, \bar{y})^{-1}[\cdot][\sum_{i=1}^{m}](\rho_i^1[\cdot]d^2(\cdot, \bar{y}))\right)[+]\left(\alpha^2(\cdot, y)^{-1}[\cdot][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(\cdot, \bar{y}))\right) \geqq 0, \tag{55}$$

where $\bar{G} = (\bar{G}_1, \bar{G}_2, \cdots, \bar{G}_p)$, $\bar{G}_j = \bar{\mu}_j[\cdot]g_j$, $j \in P$.

Then $\bar{x} = \bar{y}$.

Proof: Since (53) holds, we get from (13) that

$$[\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\bar{x}))[-][\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\bar{y})) \leqq 0. \tag{56}$$

On the other hand, since $(\bar{y}, \bar{\lambda}, \bar{\mu})$ is feasible for $(\mathrm{DMOP})_{h,\varphi}$, we have from Lemma 2.3 (d) that

$$(-1)[\cdot][\sum_{j=1}^{p}](\bar{\mu}_j[\cdot]g_j(\bar{y})) \leqq 0. \tag{57}$$

We proceed by contradiction. Suppose that $\bar{x} \neq \bar{y}$.

Assuming that condition (I) is satisfied, for weak minimum $\bar{x}$ of $(\mathrm{MOP})_{h,\varphi}$, we get from Definition 2.4 that

$$f_i(\bar{x})[-]f_i(\bar{y}) > F(\bar{x}, \bar{y}; \alpha^1(\bar{x}, \bar{y}) \otimes \nabla^* f_j(\bar{y}))[+](\rho_i^1[\cdot]d^2(\bar{x}, \bar{y})), \ i \in M,$$

and

$$(-1)[\cdot]g_i(\bar{y}) \geqq F(\bar{x}, \bar{y}; \alpha^2(\bar{x}, \bar{y}) \otimes \nabla^* g_j(\bar{y}))[+](\rho_j^2[\cdot]d^2(\bar{x}, \bar{y})), \ j \in P.$$

Since $\bar{\lambda} \geq 0$ and $\bar{\mu} \geqq 0$, by the similar argument as that of Theorem 3.1 we can get

$$[\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\bar{x}))[-][\sum_{i=1}^{m}](\bar{\lambda}_i[\cdot]f_i(\bar{y})) >$$
$$\alpha^1(\bar{x}, \bar{y})[\cdot][\sum_{i=1}^{m}]F(\bar{x}, \bar{y}; \bar{\lambda}_i \otimes \nabla^* f_i(\bar{y}))[+][\sum_{i=1}^{m}]\left((\bar{\lambda}_i \rho_i^1)[\cdot]d^2(\bar{x}, \bar{y})\right),$$

and

$$(-1)[\cdot][\sum_{j=1}^{p}](\bar{\mu}_j[\cdot]g_i(\bar{y})) \geqq$$
$$\alpha^2(\bar{x}, \bar{y})[\cdot][\sum_{j=1}^{p}]F(\bar{x}, \bar{y}; \bar{\mu}_j \otimes \nabla^* g_j(\bar{y}))[+][\sum_{j=1}^{p}]\left((\bar{\mu}_j \rho_j^2)[\cdot]d^2(\bar{x}, \bar{y})\right).$$

From (56) and (57), the above two inequalities give that

$$\alpha^1(\bar{x},\bar{y})[\cdot][\sum_{i=1}^{m}]\,F(\bar{x},\bar{y};\bar{\lambda}_i\otimes\nabla^* f_i(\bar{y}))[+][\sum_{i=1}^{m}]\big((\bar{\lambda}_i\rho_i^1)[\cdot]d^2(\bar{x},\bar{y})\big)<0,$$

and

$$\alpha^2(\bar{x},\bar{y})[\cdot][\sum_{j=1}^{p}]\,F(\bar{x},\bar{y};\bar{\mu}_j\otimes\nabla^* g_j(\bar{y}))[+][\sum_{j=1}^{p}]\big((\bar{\mu}_j\rho_j^2)[\cdot]d^2(\bar{x},\bar{y})\big)\leqq 0.$$

The rest of proof is similar to that of Theorem 3.1.

Suppose that condition (II) holds. For $\bar{x}\in F$, we get from Definition 2.8 that

$$\begin{gathered}[\sum_{i=1}^{m}]\,F(\bar{x},\bar{y};\alpha^1(\bar{x},\bar{y})\otimes\nabla^*(\bar{\lambda}_i[\cdot]f_j)(\bar{y}))[+][\sum_{i=1}^{m}]\,(\rho_i^1[\cdot]d^2(\bar{x},\bar{y}))\geqq 0\Rightarrow\\ [\sum_{i=1}^{m}]\,(\bar{\lambda}_i[\cdot]f_j(\bar{x}))>[\sum_{i=1}^{m}]\,(\bar{\lambda}_i[\cdot]f_j(\bar{y})),\end{gathered}\tag{58}$$

and

$$\begin{gathered}(-1)[\cdot][\sum_{j=1}^{p}](\bar{\mu}_j[\cdot]g_j(\bar{y}))\leqq 0\Rightarrow\\ \Big[\sum_{j=1}^{p}F(\bar{x},\bar{y};\alpha^2(\bar{x},\bar{y})\otimes\nabla^*(\bar{\mu}_j[\cdot]g_j(\bar{y}))[+][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(\bar{x},\bar{y}))\leqq 0.\end{gathered}\tag{59}$$

By Lemma 2.2 (b) and (h,φ)-sublinearity of F, (58) and (59) give that

$$\begin{gathered}\alpha^1(\bar{x},\bar{y})[\cdot][\sum_{i=1}^{m}]\,F(\bar{x},\bar{y};\bar{\lambda}_i\otimes\nabla^* f_j(\bar{y}))[+][\sum_{i=1}^{m}]\,(\rho_i^1[\cdot]d^2(\bar{x},\bar{y}))\geqq 0\Rightarrow\\ [\sum_{i=1}^{m}]\,(\bar{\lambda}_i[\cdot]f_i(\bar{x}))>[\sum_{i=1}^{m}]\,(\bar{\lambda}_i[\cdot]f_j(\bar{y})),\end{gathered}\tag{60}$$

and

$$\begin{gathered}(-1)[\cdot][\sum_{j=1}^{p}](\bar{\mu}_j[\cdot]g_j(\bar{y}))\leqq 0\Rightarrow\\ \alpha^2(\bar{x},\bar{y})[\sum_{j=1}^{p}]F(\bar{x},\bar{y};\bar{\mu}_j\otimes\nabla^* g_j(\bar{y}))[+][\sum_{j=1}^{p}](\rho_j^2[\cdot]d^2(\bar{x},\bar{y}))\leqq 0.\end{gathered}\tag{61}$$

From (53) and (57), we get that

$$\alpha^1(\bar{x},\bar{y})[\cdot][\sum_{i=1}^{m}]\,F(\bar{x},\bar{y};\bar{\lambda}_i\otimes\nabla^* f_i(\bar{y}))[+][\sum_{i=1}^{m}]\big(\rho_i^1[\cdot]d^2(\bar{x},\bar{y})\big)<0,$$

and

$$\alpha^2(\bar{x},\bar{y})[\cdot][\sum_{j=1}^{p}]\,F(\bar{x},\bar{y};\bar{\mu}_j\otimes\nabla^* g_j(\bar{y}))[+][\sum_{j=1}^{p}]\big(\rho_j^2[\cdot]d^2(\bar{x},\bar{y})\big)\leqq 0.$$

Now, We can complete the proof by following the lines of Theorem 3.2.

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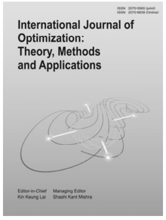

Chance Constrained Programming Problem under Different Fuzzy Distributions

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Abstract. This paper develops a solution method for Chance Constrained Programming problem, where the random variable in the right hand side of the chance constraints follows different fuzzy distributions. The methodology is verified through numerical examples.

Keywords: fuzzy random variable, Fuzzy Chance Constrained Programming problem, fuzzy uniform distribution, fuzzy exponential distribution, fuzzy conditional probability.

1 Introduction

Fuzzy Chance Constrained Programming(FCCP) problem is a chance constrained programming problem in the presence of ambiguous information. Many researchers like Liu[7], Luhandjula[9], [10] have derived different methods to solve such type of problems. All these techniques involve fuzziness and randomness in different scenarios. In this paper randomness and fuzziness are considered under one roof in the form of fuzzy random variable. The concept of fuzzy random variable was first introduced by Zadeh [14] and further developed by Kwakernaak [6], Kratschmer[5] according to different requirements of measurability. Buckley ([1], [2], [3]) has defined fuzzy probability using fuzzy numbers as parameters in probability density function and probability mass function. These fuzzy numbers are obtained from the set of confidence interval. The approach of fuzzy probability theory by Buckley is different from his predecessors and also comfortable for computational point

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of view. Nanda, Panda and Dash[11, 12] have discussed the deterministic equivalent of FCCP in the presence of normally distributed fuzzy random variable in different scenarios. This paper is different from previous works due to the presence of fuzzy random variable following other types of distributions and conditional fuzzy probability distribution inside the chance constraint.

A crisp Chance Constrained Programming(CCP) problem is of the form,

$$(CCP): \qquad Minimize \sum_{j=1}^{n} c_j x_j$$

$$Subject\ \ to\ \ P(\sum_{j=1}^{n} a_{ij} x_j \le b_i) \ge p_i$$

$$\sum_{j=1}^{n} b_{kj} x_j \ge h_k,\ \ x_j \ge 0$$

$$i = 1, 2, ...m,\ k = 1, 2, ...K,\ \ 0 \le p_i \le 1.$$

where at least one of c_j, a_{ij}, b_i is a random variable. In the present work, b_i follows fuzzy probability distribution and fuzzy conditional distribution.

The paper is organized in the following sections. Some prerequisites regarding the concept of fuzzy random variable, fuzzy conditional probability and some results from previous works are discussed in Section-2. In Section-3.1, a Fuzzy Chance Constrained Programming(FCCP) problem is considered in which the right hand side of at least one of the constraint (b_i) is a fuzzy random variable following different distributions. Section 3.2 deals with FCCP in the presence of fuzzy conditional distribution. In both the sections, methodologies are developed to convert the FCCP to an equivalent crisp model. The methodology is justified through numerical examples. The last section presents further research scope with some concluding remarks.

2 Preliminaries

Fuzzy Number:

A fuzzy number $\tilde{A}$ is a convex normalized fuzzy set $\tilde{A}$ of the real line *R*, with membership function $\mu_A : R \to [0,1]$, satisfying the following conditions.

1. There exists exactly one interval $I \in R$ such that $\mu_{\tilde{A}}(x) = 1$, for all $x \in I$.
2. The membership function $\mu_{\tilde{A}}(x)$ is piecewise continuous.

A fuzzy number $\tilde{A} = \langle a / b / c \rangle$ is said to be triangular if its membership function is strictly increasing in the interval (a, b) and strictly decreasing in (b, c) and one at b, where b is core, $b - a$ is left spread and $c - b$ is right spread. $\tilde{A}$ is a linear triangular fuzzy number if the membership function is linear. α -cut of the fuzzy number $\tilde{A}$ is the set $\{x \mid \mu_{\tilde{A}}(x) \ge \alpha\}$ for $0 \le \alpha \le 1$ and denoted by $\tilde{A}[\alpha]$. A fuzzy number $\tilde{A}$ is said to be positive if it's membership function $\mu_{\tilde{A}}(x) = 0$, for all $x \le 0$. If $\tilde{A}$ is a triangular fuzzy number then this definition is equivalent to following form. Let

$\tilde{A}[\alpha]=[a_*(\alpha),a^*(\alpha)]$ be the α -cut of a triangular fuzzy number $\tilde{A}$ for $0\le\alpha\le1$. $\tilde{A}$ is said to be positive if $a_*(0)>0$. Throughout this paper all the fuzzy numbers are considered to be linear triangular.

Inequalities[1,13]

Let $\tilde{A}=\langle a_1/a_2/a_3\rangle$ be a triangular fuzzy number, r and s be two real numbers, then $r\le\tilde{A}\le s$ if $a_1\ge r$ and $a_3\le s$. The set of fuzzy numbers is a partial ordered set. Different types of partial order relations exist in the theory of fuzzy sets using α -cut and defuzzification method. Nanda and Kar [13] defined a partial order relation $\succeq$ between two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ using α -cuts $\tilde{A}[\alpha]$ and $\tilde{B}[\alpha]$ respectively as $\tilde{A}\succeq\tilde{B}$ iff $a\ge b$, for all $a\in\tilde{A}[\alpha]$ and $b\in\tilde{B}[\alpha]$, for each $\alpha\in[0,1]$. This definition of $\tilde{A}\succeq\tilde{B}$ is equivalent to the following form. Let $\tilde{A}[\alpha]=[a_*(\alpha),a^*(\alpha)]$ and $\tilde{B}[\alpha]=[b_*(\alpha),b^*(\alpha)]$. So $\tilde{A}\succeq\tilde{B}$ iff $a_*(\alpha)\ge b^*(\alpha)$ for each $\alpha\in[0,1]$. The advantage of this type of partial order relation is that it reduces mathematical computation.

Fuzzy Probability[1,2]

Let X be a continuous random variable with probability density function $f(x,\theta)$, where θ is a parameter describing the probability density function. Suppose θ is uncertain and estimated from confidence interval. Then θ can be generated as a fuzzy number, $\tilde{\theta}$ (For detail one may refer [1,2]). We denote $\tilde{X}$ for continuous random variable with fuzzy parameter $\tilde{\theta}$ and $\tilde{P}$ as fuzzy probability. Then $\tilde{X}$ is said to be a continuous fuzzy random variable with density $f(x,\tilde{\theta})$ such that $\int_{-\infty}^{\infty}f(x,\theta)dx=1$ for $\theta\in\tilde{\theta}[\alpha]$. $\tilde{X}\le x$ is an event which means the random variable X takes values less than or equal to x where the probability of this event involve some vagueness and denoted by $\tilde{P}(\tilde{X}\le x)$. This is a fuzzy number $\tilde{p}$, $0\le\tilde{p}\le1$, where the inequalities are defined as above. That is if $\tilde{p}=\langle p_1/p_2/p_3\rangle$ is a fuzzy number then $p_1\ge0, p_3\le1$. Let $A=[c,d]$ be an event. Then the probability of the event A of the continuous fuzzy random variable $\tilde{X}$ is a fuzzy number whose α -cut is denoted by $\tilde{P}(c\le\tilde{X}\le d)[\alpha]$, where

$$\tilde{P}(c\le\tilde{X}\le d)[\alpha]=\left\{\int_c^d f(x,\theta)dx \mid \theta\in\tilde{\theta}[\alpha]\right\}=[p_*(\alpha),p^*(\alpha)]\ \ (say)$$

where $p_*(\alpha)$ and $p^*(\alpha)$ are the optimal objective values of the following optimization problems,

$$\begin{aligned}&\textit{Minimize} && \int_c^d f(x,\theta)dx\\ &\textit{Subject to} && \int_{-\infty}^{\infty}f(x,\theta)dx=1,\quad \theta\in\tilde{\theta}[\alpha]\qquad\text{and}\\ &\textit{Maximize} && \int_c^d f(x,\theta)dx\\ &\textit{Subject to} && \int_{-\infty}^{\infty}f(x,\theta)dx=1,\quad \theta\in\tilde{\theta}[\alpha]\end{aligned}$$

respectively, $\tilde{\theta}[\alpha]$ is the α -cut of the fuzzy parameter θ. Hence a fuzzy random variable is a random variable whose parameter is fuzzy number.

Fuzzy Conditional Distribution[1]

Let X and Y are two random variables with joint density function $f(x,y,\theta)$ and marginal density of Y is $f_y(y,\theta)$. Due to the presence of fuzzy parameter in joint density function and marginal density function they will be treated as fuzzy numbers. In that case, we may denote the fuzzy random variables as $\tilde{X},\tilde{Y}$ with joint density $f(x,y,\tilde{\theta})$ and marginal density of Y is $f_y(y,\tilde{\theta})$. Due to the presence of fuzzy parameters in the density function the conditional probability of $\tilde{X}$ given $\tilde{Y}$ at y will be a fuzzy number . The α cut of this conditional probability of $\tilde{X}\geq x$ where $\tilde{Y}=y$ is

$$\begin{aligned}\tilde{P}(\tilde{X}\geq x\,|\,\tilde{Y}=y)[\alpha]&=\left\{\int_x^{\infty}\tfrac{f(x,y,\theta)}{f_y(y,\theta)}dx\,|\,\theta\in\tilde{\theta}[\alpha],\int_{-\infty}^{\infty}\tfrac{f(x,y,\theta)}{f_y(y,\theta)}dx=1\right\}\\&=[p_*(\alpha),p^*(\alpha)]\end{aligned}$$

where $p_*(\alpha)$ and $p^*(\alpha)$ are the optimal objective values of the following optimization problems,

$$\begin{aligned}&\textit{Minimize} && \int_x^{\infty}\tfrac{f(x,y,\theta)}{f_y(y,\theta)}dx\\&\textit{Subject to} && \int_{-\infty}^{\infty}\tfrac{f(x,y,\theta)}{f_y(y,\theta)}dx=1\\& && \theta\in\tilde{\theta}[\alpha],\ f_y(y,\theta)\in f_y(y,\tilde{\theta})[\alpha]\qquad\text{and}\end{aligned}$$

$$\begin{aligned}&\textit{Maximize} && \int_x^{\infty}\tfrac{f(x,y,\theta)}{f_y(y;\theta)}dx\\&\textit{Subject to} && \int_{-\infty}^{\infty}\tfrac{f(x,y,\theta)}{f_y(y,\theta)}dx=1\\& && \theta\in\tilde{\theta}[\alpha],\ f_y(y,\theta)\in f_y(y,\tilde{\theta})[\alpha]\end{aligned}$$

respectively. The α cut of the conditional probability of $\tilde{Y}\geq y$ where $\tilde{X}=x$ can be determined in a similar way.

Note: It is not hard to see that α -cut of the probability of a fuzzy random variable and conditional probability are closed intervals. For more details on fuzzy random variable, conditional distribution and their physical significance, the readers may see Buckley[1,2,3]. Throughout this paper we denote $\tilde{P}(\tilde{X}\leq x_i)[\alpha]=[p_*^i(\alpha),p^{i*}(\alpha)]$.

3 Problem Formulation and Methodology

Consider a Chance Constrained Programming(CCP) problem, discussed in Section 1. A Fuzzy Chance Constrained Programming(FCCP) problem as a CCP problem where at least one of c_j,a_{ij},b_i is a fuzzy random variable(FRV) and p_i is a fuzzy number. In this paper, the presence of FRV is considered only in the right hand side of the constraints in two types of situations. In Section 3.1, $\tilde{b}_i$ is considered as fuzzy random variables following different types of fuzzy probability distributions and in Section 3.2, $\tilde{b}_i$ is associated with fuzzy conditional probability distribution.

3.1 $\tilde{b}_i$ Is A Fuzzy Random Variable

Let $\tilde{b}_i$ be a fuzzy random variable following different types of fuzzy probability distributions. In that case the CCP takes the following form, which we denote by $FCCP$.

$$(FCCP): \qquad Minimize \sum_{j=1}^{n} c_j x_j$$

$$Subject\ \ to\ \ \tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i) \succeq \tilde{p}_i$$

$$\sum_{j=1}^{n} b_{kj}x_j \ge h_k, \quad x_j \ge 0$$

$$i = 1,2,...m,\ k = 1,2,...K,\ 0 \le p_i \le 1,\ p_i \in \tilde{p}_i[\alpha]$$

where $\tilde{b}_i$ are fuzzy random variables with density function $f(b_i, \tilde{\theta})$, $\tilde{\theta}$ is a fuzzy parameter and $\tilde{p}_i$ are fuzzy numbers, $\succeq$ is greater than or equal to in fuzzy sense, which is a fuzzy partial order relation. Let $\sum_{j=1}^{n} a_{ij}x_j = u_i$. $\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i$ is a fuzzy event. $\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i) = \tilde{P}(u_i \le \tilde{b}_i)$ is the probability of a fuzzy event which is a fuzzy number. Its α-cut is the set,

$$\tilde{P}(u_i \le \tilde{b}_i)[\alpha] = \left\{ \int_{u_i}^{\infty} f(b_i, \theta) db_i \mid \theta \in \tilde{\theta}[\alpha] \right\}$$

$$= [p_*^i(\alpha), p^{*i}(\alpha)] (\text{say})$$

Let $\tilde{p}_i[\alpha] = [p_{i*}(\alpha), p_i^*(\alpha)]$ denote the α-cut of the fuzzy number $\tilde{p}_i$. In the fuzzy constraint $\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i) \succeq \tilde{p}_i$, both sides of the inequality are fuzzy numbers. $\succeq$ is a partial order relation between two fuzzy numbers. This fuzzy inequality can be converted to a total order relation using the α-cuts of both fuzzy numbers as described in Section 2.
$\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i) \succeq \tilde{p}_i$ iff $x \ge y$ for all $x \in [p_*^i(\alpha), p^{*i}(\alpha)]$, $y \in [p_{i*}(\alpha), p_i^*(\alpha)]$ and each $\alpha \in [0,1]$, which is equivalent to

$$p_*^i(\alpha) \ge p_i^*(\alpha) \tag{3.1}$$

where $p_*^i(\alpha)$ is the optimal objective value of the following optimization problem.

$$Minimize \qquad \int_{u_i}^{\infty} f(b_i, \theta) db_i$$

$$subject\ to \qquad \theta \in \tilde{\theta}[\alpha]$$

$$\int_{-\infty}^{\infty} f(b_i, \theta) db_i = 1$$

Depending upon $\tilde{b}_i$, (3.1) takes different forms which can be described in following cases.
Case-I ($\tilde{b}_i$ follows fuzzy uniform distribution [1,2]):
Let X be a uniformly distributed random variable on $[a,b]$ with density function

$$f(x;a,b) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b, \\ 0 & \text{otherwise.} \end{cases}$$

If the random variable X takes the values in a domain where the end points are not certain, in particular if a and b are fuzzy numbers and denoted by $\tilde{a}$ and $\tilde{b}$, in that case, X is said to be a uniformly distributed fuzzy random variable and denoted by $\tilde{X}$ (for details see[1]). So probability of an event $[c,d]$ denoted by $\tilde{P}(c \le \tilde{X} \le d)$ is the fuzzy probability of $\tilde{X}$ on $[c,d]$, which is a fuzzy number and its α-cut denoted by $\tilde{P}(c \le \tilde{X} \le d)[\alpha]$ is the set

$$\left\{\int_c^d f(x,a,b)dx \mid a \in \tilde{a}[\alpha], b \in \tilde{b}[\alpha]\right\}.$$

In FCCP if $\tilde{b}_i$ is an uniformly distributed FRV on $[\tilde{\beta}_i, \tilde{\gamma}_i]$, where $\tilde{\beta}_i, \tilde{\gamma}_i$ are uncertain and treated as positive fuzzy numbers then α-cut of $\tilde{P}(\sum_{j=1}^n a_{ij}x_j \le \tilde{b}_i)$ is

$$\begin{aligned}\tilde{P}(\sum_{j=1}^n a_{ij}x_j \le \tilde{b}_i)[\alpha] &= \left\{\int_{u_i}^{\gamma_i} \frac{1}{\gamma_i - \beta_i} db_i \mid \beta_i \in \tilde{\beta}_i[\alpha], \gamma_i \in \tilde{\gamma}_i(\alpha)\right\} \\ &= \left\{\frac{\gamma_i - u_i}{\gamma_i - \beta_i} \mid \beta_i \in \tilde{\beta}_i[\alpha], \gamma_i \in \tilde{\gamma}_i(\alpha)\right\} \\ &= [p^i{}_*(\alpha), p^{i*}(\alpha)]\end{aligned}$$

where $u_i = \sum_{j=1}^n a_{ij}x_j$, $\beta_i \le u_i \le \gamma_i$ for $\beta_i \in \tilde{\beta}_i[\alpha], \gamma_i \in \tilde{\gamma}_i(\alpha)$ ($f(u_i, \beta_i, \gamma_i) = 0$ in other cases). $p^i{}_*(\alpha)$ and $p^{i*}(\alpha)$ are optimal objective values of the following optimization problems,

$$\begin{aligned}(P_1): \quad & \textit{Minimize} && \frac{\gamma_i - u_i}{\gamma_i - \beta_i} \\ & \textit{subject to} && \gamma_i \in \tilde{\gamma}_i[\alpha], \\ &&& \beta_i \in \tilde{\beta}_i[\alpha]\end{aligned}$$

$$\begin{aligned}(P_2): \quad & \textit{Maximize} && \frac{\gamma_i - u_i}{\gamma_i - \beta_i} \\ & \textit{subject to} && \gamma_i \in \tilde{\gamma}_i[\alpha] \\ &&& \beta_i \in \tilde{\beta}_i[\alpha]\end{aligned}$$

respectively.

Theorem 3.1 *If $\tilde{b}_i$ is an uniformly distributed FRV on $[\tilde{\beta}_i, \tilde{\gamma}_i]$, then 3.1 is equivalent to the system of following inequations.*

$$\sum_{j=1}^n a_{ij}x_j \le (1 - p_i^*(\alpha))\gamma_i^*(\alpha) + p_i^*(\alpha)\beta_{i*}(\alpha),$$

$$\sum_{j=1}^n a_{ij}x_j \le (1 - p_i^*(\alpha))\gamma_i^*(\alpha) + p_i^*(\alpha)\beta_i^*(\alpha),$$

$$\sum_{j=1}^n a_{ij}x_j \le (1 - p_i^*(\alpha))\gamma_{i*}(\alpha) + p_i^*(\alpha)\beta_i^*(\alpha),$$

$$\sum_{j=1}^{n} a_{ij}x_j \le (1-p_i^*(\alpha))\gamma_{i*}(\alpha)+p_i^*(\alpha)\beta_{i*}(\alpha)$$

where $\alpha \in [0,1]$, $\tilde{\beta}_i$ and $\tilde{\gamma}_i$ are positive fuzzy numbers, $\tilde{\gamma}_i[\alpha]=[\gamma_{i*}(\alpha),\gamma_i^*(\alpha)], \tilde{\beta}_i[\alpha]=[\beta_{i*}(\alpha),\beta_i^*(\alpha)]$, $\tilde{p}_i[\alpha]=[p_{i*}(\alpha),p_i^*(\alpha)]$

Proof: To get the equivalent form of 3.1, it is necessary to solve only (P_1) which takes the following form

$$(P_1): \qquad \textit{Minimize} \quad \frac{\gamma_i-u_i}{\gamma_i-\beta_i}$$
$$\textit{subject to} \quad \beta_{i*}(\alpha)\le\beta_i\le\beta_i^*(\alpha)$$
$$\gamma_{i*}(\alpha)\le\gamma_i\le\gamma_i^*(\alpha)$$

(P_1) is a linear fractional programming problem for each $0\le\alpha\le1$, where $\gamma_i>\beta_i$. Let $\frac{1}{\gamma_i-\beta_i}=\delta_i$, $(\delta_i>0)$, $\gamma_i\delta_i=z_i^1$, $\beta_i\delta_i=z_i^2$. Then (P_1) becomes a linear programming problem in three variables z_i^1, z_i^2 and δ_i. Putting $z_i^2=z_i^1-1$, it reduces to the following linear programming problem $(P_1)'$ in two variables.

$$(P_1)': \quad \underset{z_i^1,\delta_i\in S}{\textit{Minimize}} \; z_i^1-u_i\delta_i$$

where

$$S=\{z_i^1\le\delta_i\gamma_i^*(\alpha)\}\cap\{z_i^1\ge\delta_i\gamma_{i*}(\alpha)\}\cap\{z_i^1\le1+\delta_i\beta_i^*(\alpha)\}\cap\{z_i^1\ge1+\delta_i\beta_{i*}(\alpha)\}$$

Here $\gamma_i^*(\alpha)>\gamma_{i*}(\alpha)>\beta_i^*(\alpha)>\beta_{i*}(\alpha)$. So the slopes of L_1,L_2,L_3 and L_4 are in increasing order. Hence S is convex polyhedron ABCD as shown in the Fig. 1, where

$$A=\left(\frac{\gamma_i^*(\alpha)}{\gamma_i^*(\alpha)-\beta_{i*}(\alpha)},\frac{1}{\gamma_i^*(\alpha)-\beta_{i*}(\alpha)}\right), B=\left(\frac{\gamma_i^*(\alpha)}{\gamma_i^*(\alpha)-\beta_i^*(\alpha)},\frac{1}{\gamma_i^*(\alpha)-\beta_i^*(\alpha)}\right)$$

$$C=\left(\frac{\gamma_{i*}(\alpha)}{\gamma_{i*}(\alpha)-\beta_i^*(\alpha)},\frac{1}{\gamma_{i*}(\alpha)-\beta_i^*(\alpha)}\right), D=\left(\frac{\gamma_{i*}(\alpha)}{\gamma_{i*}(\alpha)-\beta_{i*}(\alpha)},\frac{1}{\gamma_{i*}(\alpha)-\beta_{i*}(\alpha)}\right)$$

The objective function $z_i^1-u_i\delta_i$ is a linear function and u_i is positive. So the minimum of the objective function will exist at the vertices of the convex polyhedron. Hence 3.1 is equivalent to

$$Min\left\{\frac{\gamma_i^*(\alpha)-u_i}{\gamma_i^*(\alpha)-\beta_{i*}(\alpha)},\frac{\gamma_i^*(\alpha)-u_i}{\gamma_i^*(\alpha)-\beta_i^*(\alpha)},\frac{\gamma_{i*}(\alpha)-u_i}{\gamma_{i*}(\alpha)-\beta_i^*(\alpha)},\frac{\gamma_{i*}(\alpha)-u_i}{\gamma_{i*}(\alpha)-\beta_{i*}(\alpha)}\right\}\ge p_i^*(\alpha).$$

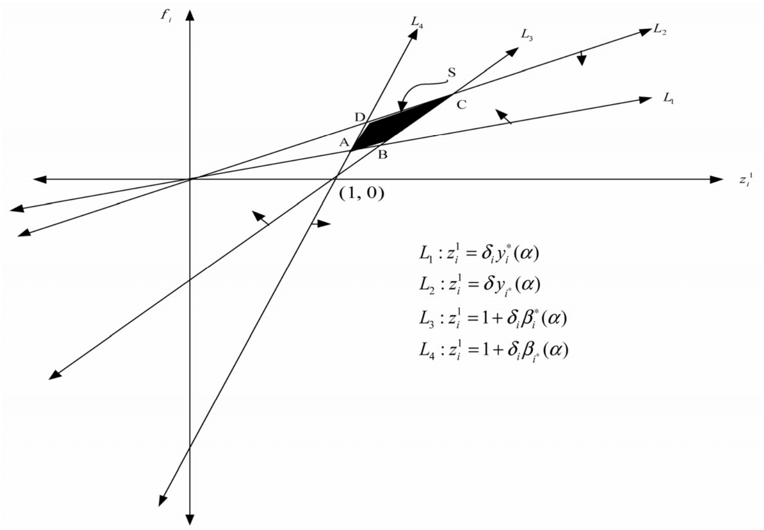

Fig. 1. Feasible region of P_1'

Hence

$$\begin{aligned}
\gamma_i^*(\alpha) - u_i &\geq (\gamma_i^*(\alpha) - \beta_{i*}(\alpha))\, p_i^*(\alpha) \\
\gamma_i^*(\alpha) - u_i &\geq (\gamma_i^*(\alpha) - \beta_i^*(\alpha))\, p_i^*(\alpha) \\
\gamma_{i*}(\alpha) - u_i &\geq (\gamma_{i*}(\alpha) - \beta_i^*(\alpha))\, p_i^*(\alpha) \\
\gamma_{i*}(\alpha) - u_i &\geq (\gamma_{i*}(\alpha) - \beta_{i*}(\alpha))\, p_i^*(\alpha)
\end{aligned}$$

The above inequalities reduce to the system of inequalities of the theorem.

In some real life situations $\tilde{X}$ may be uniformly distributed over an interval where one end point is uncertain. For example[1](page 100-101), consider that the customers arrive randomly at a certain shop. Given that one customer arrived during a particular time T-minute period, let X be the time within the T minutes that the customer arrived, where T is uncertain and approximately equal to γ. Let "approximately equal to γ" be a fuzzy number denoted by $\tilde{\gamma}$. Then the random variable X can be treated as a fuzzy random variable which is uniformly distributed over the uncertain domain $[0, \tilde{\gamma}]$. We denote this FRV as $\tilde{X}$. In general if $\tilde{X}$ is a uniformly distributed FRV over an interval $[\beta, \tilde{\gamma}]$, where one end point $\tilde{\gamma}$ is uncertain, then α-cut of $\tilde{P}(c \leq \tilde{X} \leq d)$ is

$$\tilde{P}(c \leq \tilde{X} \leq d)[\alpha] = \left\{ \int_c^d \frac{1}{\gamma - \beta} dx \mid \gamma \in \tilde{\gamma}[\alpha] \right\}, \quad \beta \leq c < d \leq \gamma$$

In the FCCP if $\tilde{b}_i$ is uniformly distributed FRV on $[\beta, \tilde{\gamma}]$, where $\tilde{\gamma}$ is a positive fuzzy number then the following result hold.

Corollary 3.2 *If $\tilde{b}_i$ is an uniformly distributed FRV on $[\beta_i, \tilde{\gamma}_i]$, where $\tilde{\gamma}_i$ is a positive fuzzy number, then 3.1 is equivalent to*

$$u_i \le (1-p_i^*(\alpha))\gamma_{i*}(\alpha)+p_i^*(\alpha)\beta_i,\ \ \alpha\in[0,1]$$

where $\tilde{\gamma}_i[\alpha]=[\gamma_{i*}(\alpha),\gamma_i^*(\alpha)]$.

Proof: The proof is easy and straight forward.

$$\begin{aligned}\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i)[\alpha] &= \left\{\int_{u_i}^{\gamma_i}\frac{1}{\gamma_i-\beta_i}db_i \mid \gamma_i\in\tilde{\gamma}_i(\alpha)\right\}\\ &=\left\{\frac{\gamma_i-u_i}{\gamma_i-\beta_i}\mid \gamma_i\in\tilde{\gamma}_i(\alpha)\right\}\\ &=[p^i{}_*(\alpha),p^{i*}(\alpha)]\end{aligned}$$

where $\sum_{j=1}^{n} a_{ij}x_j = u_i$, $p^i{}_*(\alpha)$ and $p^{i*}(\alpha)$ are the optimal objective values of the following optimization problems,

$$\begin{aligned}&\textit{Minimize}\ \frac{\gamma_i-u_i}{\gamma_i-\beta_i}\\ &\textit{subject to}\ \gamma_{i*}(\alpha)\le\gamma_i\le\gamma_i^*(\alpha),\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\textit{Maximize}\ \frac{\gamma_i-u_i}{\gamma_i-\beta_i}\\ &\textit{subject to}\ \gamma_{i*}(\alpha)\le\gamma_i\le\gamma_i^*(\alpha),\end{aligned} \tag{3.3}$$

respectively. It is required to solve the first problem only. Here the objective function is an increasing function of γ_i. Hence the minimum takes place at $\gamma_{i*}(\alpha)$. It is easy to deduce 3.1 to the form $u_i \le (1-p_i^*(\alpha))\gamma_{i*}(\alpha)+p_i^*(\alpha)\beta_i$

Case-II ($\tilde{b}_i$ follows fuzzy exponential distribution[2])

Let X be an exponentially distributed random variable with mean $\frac{1}{\lambda}$ and density function

$$f(x;\lambda)=\begin{cases}\lambda e^{-\lambda x} & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$$

Let λ be estimated from confidence interval and generated to a fuzzy number, denoted by $\tilde{\lambda}$. Then the random variable X will be a fuzzy exponentially distributed random variable, denoted by $\tilde{X}$. So probability of an event $[c,d]$ denoted by $\tilde{P}(c\le\tilde{X}\le d)$ is the fuzzy probability of $\tilde{X}$ on $[c,d]$, which is a fuzzy number. It's α-cut is

$$\tilde{P}(c\le\tilde{X}\le d)[\alpha]=\left\{\int_c^d \lambda e^{-\lambda x}dx \mid \lambda\in\tilde{\lambda}[\alpha]\right\}$$

Theorem 3.3 *If $\tilde{b}_i$ is an exponentially distributed FRV then $\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \preceq \tilde{b}_i)\succeq \tilde{p}_i$ is equivalent to $\ln p_i^*(\alpha)+\lambda_i^*(\alpha)\sum_{j=1}^{n} a_{ij}x_j \le 0$ for $\alpha\in[0,1]$, where $\tilde{\lambda}_i[\alpha]=[\lambda_{i*}(\alpha),\lambda_i^*(\alpha)]$*

Proof:

The α-cut of the fuzzy probability $\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \preceq \tilde{b}_i)$ is

$$\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \preceq \tilde{b}_i)[\alpha] = \left\{\int_{u_i}^{\infty} \lambda_i e^{-\lambda_i b_i} db_i \mid \lambda_i \in \tilde{\lambda}_i[\alpha]\right\}$$
$$= \left\{e^{-\lambda_i u_i} \mid \lambda_{i*}(\alpha) \le \lambda_i \le \lambda_i^*(\alpha)\right\}$$
$$= [\, p^i{}_*(\alpha), p^{i*}(\alpha)\ ]$$

Where $u_i = \sum_{j=1}^{n} a_{ij}x_j$, $p^i{}_*(\alpha)$ and $p^{i*}(\alpha)$ are the optimal objective values of the optimization problems

$$\underset{\lambda_{i*}(\alpha) \le \lambda_i \le \lambda_i^*(\alpha)}{Minimize} \; e^{-\lambda_i u_i} \quad \text{and} \quad \underset{\lambda_{i*}(\alpha) \le \lambda_i \le \lambda_i^*(\alpha)}{Maximize} \; e^{-\lambda_i u_i}$$

respectively. $e^{-\lambda_i u_i}$ is a decreasing function of λ_i. Hence $e^{-\lambda_i u_i}$ attains it's minimum at $\lambda_i^*(\alpha)$ and maximum at $\lambda_{i*}(\alpha)$. So

$$\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \preceq \tilde{b}_i)[\alpha] = [e^{-\lambda_i^*(\alpha)u_i}, e^{-\lambda_{i*}(\alpha)u_i}] \tag{3.4}$$

Using 3.1 and 3.2 the deterministic equivalent of the fuzzy chance constraint $\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \preceq \tilde{b}_i) \succeq \tilde{p}_i$ becomes $\ln p_i^*(\alpha) + \lambda_i^*(\alpha)\sum_{j=1}^{n} a_{ij}x_j \le 0$.

The inequality 3.1 can be converted to some deterministic equivalent form in case $\tilde{b}_i$ follows other type of fuzzy probability distributions. In general the deterministic equivalent of the fuzzy chance constrained programming problem becomes

$(FCCP_1)$: *Minimize* $\sum_{j=1}^{n} c_j x_j$

Subject to Deterministic equivalent of fuzzy chance constraint

$$\sum_{j=1}^{n} b_{kj}x_j \ge h_k, x_j \ge 0$$

$$\alpha \in [0,1], i = 1,2,...m, k = 1,2,...K, x_j \in R.$$

The deterministic equivalent of the FCCP is a non linear programming problem which can be handled by writing a program using any numerical optimization technique or softwares like Lingo, CPLEX etc. The above methodology is justified in the following numerical example.

Example 3.4 *Consider the following* $(FCCP)$ *problem*

$$Minimize \;\; x + 2y$$
$$Subject \;\; to \;\; \tilde{P}(3x - 4y \preceq \tilde{b}) \succeq \widetilde{0.4}$$
$$x + y \ge 1, 5x + 3y \le 60, \;\; x \ge 0, y \ge 0$$

wher $\tilde{b}$ *is a uniformly distributed FRV on* $[\tilde{2},\widetilde{12}]$, $\widetilde{0.4} = \langle 0.3/0.4/0.5\rangle$, $\tilde{2} = \langle 1/2/4\rangle$, $12 = \langle 8/12/14\rangle$ *are linear triangular fuzzy numbers.*

Solution

Using Theorem-3.1 and inequality 3.1, the deterministic equivalent of this problem is,

$(FCCP_1)$:

$$\begin{aligned} &\textit{Minimize} && x+2y \\ &\textit{Subject to} && 3x-4y \le (0.5+0.1\alpha)(14-2\alpha)+(0.5-0.1\alpha)(1+\alpha) \\ &&& 3x-4y \le (0.5+0.1\alpha)(14-2\alpha)+(0.5-0.1\alpha)(4-\alpha) \\ &&& 3x-4y \le (0.5+0.1\alpha)(8+4\alpha)+(0.5-0.1\alpha)(4-\alpha) \\ &&& 3x-4y \le (0.5+0.1\alpha)(8+4\alpha)+(0.5-0.1\alpha)(1+\alpha) \\ &&& x+y \ge 1,\ 5x+3y \le 60 \\ &&& x \ge 0,\ y \ge 0,\ 0 \le \alpha \le 1. \end{aligned}$$

The above problem is a nonlinear programming problem. Using Lingo package it's local optimal solution is $x = 1, y = 0$.

If $\tilde{b}$ is an exponentially distributed random variable with parameter $\tilde{\lambda} = \langle 5/7/10 \rangle$, then by Theorem 3.3 the deterministic equivalent of the fuzzy probability constraint is $e^{-(10-3\alpha)(3x-4y)} \ge 0.5-0.1\alpha$. Hence the corresponding $(FCCP_1)$, which will be a nonlinear programming problem. Using Lingo package it's local optimal solution is found to be $x = 0.5900039, y = 0.4099961$.

3.2 $\tilde{b}_i$ Is Associated with Fuzzy Conditional Distribution

Fuzzy conditional distribution is discussed in Section 2. Consider the following FCCP problem where the chance constraint is expressed in terms of fuzzy conditional distribution.

$(FCCP)'$:

$$\begin{aligned} &\textit{Minimize} && \sum_{j=1}^{n} c_j x_j \\ &\textit{Subject to} && \tilde{P}\Big(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i \,\Big|\, \sum_{j=1}^{n} c_{ij}x_j = \tilde{a}_i\Big) \succeq \tilde{p}_i \\ &&& \sum_{j=1}^{n} b_{kj}x_j \ge h_k,\ x_j \ge 0 \\ &&& i = 1,2,\cdots m,\ k = 1,2,\cdots K \end{aligned}$$

where $(\tilde{b}_i, \tilde{a}_i)$ is a jointly distributed fuzzy random variable with probability density function f $(b_i, a_i, \tilde{\theta})$ and $f_{a_i}(a_i, \tilde{\theta})$ is the fuzzy marginal density function of $\tilde{a}_i$. $\tilde{\theta}$ is a fuzzy parameter, p_i is a fuzzy number and $\succeq$ is greater than or equal to in fuzzy sense as defined in section 1. Let $\sum_{j=1}^{n} a_{ij}x_j = u_i$ and $\sum_{j=1}^{n} c_{ij}x_j = v_i$. $\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \preceq \tilde{b}_i \mid \sum_{j=1}^{n} c_{ij}x_j = \tilde{a}_i) = \tilde{P}(u_i \preceq \tilde{b}_i \mid v_i = \tilde{a}_i)$, is a fuzzy number whose α-cut is

$$\begin{aligned} \tilde{P}(u_i \preceq \tilde{b}_i \mid v_i = \tilde{a}_i)[\alpha] &= \left\{ \int_{u_i}^{\infty} \tfrac{f(b_i, a_i, \theta)}{f_{a_i}(a_i, \theta)} db_i \,\Big|\, \theta \in \tilde{\theta}[\alpha], \int_{-\infty}^{\infty} \tfrac{f(b_i, a_i, \theta)}{f_{a_i}(a_i, \theta)} db_i = 1 \right\} \\ &= [p^i{}_*(\alpha), p^{i*}(\alpha)] \end{aligned}$$

where $p^i{}_*(\alpha)$ and $p^{i*}(\alpha)$ are the optimal objective values of the following optimization problems,

$$\begin{aligned} &Minimize \int_{u_i}^{\infty} \frac{f(b_i,a_i,\theta)}{f_{a_i}(a_i,\theta)} db_i \\ &subject\ to \int_{-\infty}^{\infty} \frac{f(b_i,a_i,\theta)}{f_{a_i}(a_i,\theta)} db_i = 1 \\ &\theta_*(\alpha) \le \theta \le \theta^*(\alpha) \end{aligned}$$

and

$$\begin{aligned} &Maximize \int_{u_i}^{\infty} \frac{f(b_i,a_i,\theta)}{f_{a_i}(a_i,\theta)} db_i \\ &subject\ to \int_{-\infty}^{\infty} \frac{f(b_i,a_i,\theta)}{f_{a_i}(a_i,\theta)} db_i = 1 \\ &\theta_*(\alpha) \le \theta \le \theta^*(\alpha) \end{aligned}$$

respectively. Let $\tilde{p}_i[\alpha] = [p_{i*}(\alpha), p_i^*(\alpha)]$ denotes the α-cut of the fuzzy number $\tilde{p}_i$. So $\tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i \mid \sum_{j=1}^{n} c_{ij}x_j = \tilde{a}_i) \succeq \tilde{p}_i$ is equivalent to

$$p^i{}_*(\alpha) \ge p_i{}^*(\alpha) \tag{3.5}$$

Example 3.5 *Consider the following problem*

$$\begin{aligned} &Minimize\ \ x + 2y \\ &Subject\ \ to\ \ \tilde{P}(5x - 7y \le \tilde{b} \mid 2x - 3y = \tilde{a}) \succeq \widetilde{0.4} \\ &x + y \ge 1, 5x + 3y \le 60,\ \ x \ge 0, y \ge 0 \end{aligned}$$

where $(\tilde{b},\tilde{a})$ *is a jointly distributed fuzzy random variable with probability density function* $f(b,a,\tilde{\lambda})$, *which is defined as*

$$f(b,a,\tilde{\lambda}) = \tilde{\lambda}^2 e^{-\tilde{\lambda} b},\ \text{for}\ \ 0 < a < b$$

$\tilde{\lambda} = \langle 5/7/10 \rangle$, $\widetilde{0.4} = \langle 0.3/0.4/0.5 \rangle$ *are triangular fuzzy numbers.*

Solution

Now the α-cut of the probability $\tilde{P}(5x - 7y \le \tilde{b} \mid 2x - 3y = \tilde{a})$ is

$$\begin{aligned} \tilde{P}(5x - 7y \le \tilde{b} \mid 2x - 3y = \tilde{a})[\alpha] &= \left\{ \int_{5x-7y}^{\infty} \lambda e^{-\lambda(b-2x+3y)} db \mid \lambda \in \tilde{\lambda}[\alpha] \right\} \\ &= \left\{ e^{-\lambda(5x-7y-2x+3y)} \mid \lambda \in \tilde{\lambda}[\alpha] \right\} \\ &= [e^{-(10-3\alpha)(3x-4y)}, e^{-(5+2\alpha)(3x-4y)}] \end{aligned}$$

Hence the deterministic equivalent of the fuzzy conditional constraint $\tilde{P}(5x - 7y \le \tilde{b} \mid 2x - 3y = \tilde{a}) \succeq \widetilde{0.4}$ is equivalent to

$$e^{-(10-3\alpha)(3x-4y)} \ge 0.5 - 0.1\alpha$$

Using this, the deterministic equivalent of the given problem is

$$\begin{gathered} \textit{Minimize} \ \ x + 2y \\ \textit{Subject to} \ \ e^{-(10-3\alpha)(3x-4y)} \geq 0.5 - 0.1\alpha \\ x + y \geq 1, 5x + 3y \leq 60, \ \ x \geq 0, y \geq 0, \ \ 0 \leq \alpha \leq 1 \end{gathered}$$

The above problem is a crisp non linear programming problem. Using Lingo package it's local optimum solution is $x = 0.590039$, $y = 0.4099961$.

4 Conclusion

A Chance Constrained Programming problem in the presence of vague information can be solved only after converting it to some crisp equivalent. Different crisp equivalent formulations are possible depending upon the presence of vagueness inside the problem. In this paper the vagueness is present inside the chance constraint in the form of fuzzy random variable following different distributions. FCCP can also be defined in the presence of hybrid variable (defined by Liu[9]) in place of fuzzy random variable. Both fuzzy random variable and hybrid variable have uncertainties. But they are defined in different scenario. It is interesting to compare the crisp conversion of FCCP in the presence of hybrid variable and fuzzy random variable. This is the future scope of this research work. There are real life situations where two constraints have some correlation. In that case the concept of joint fuzzy random variable or joint hybrid variable inside the chance constraint comes into picture. Formulation of the crisp equivalent of such type of situation of a FCCP problem is another research scope of the present work.

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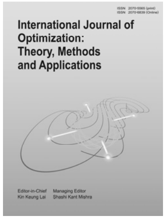

Remarks on M. A. Hanson's Paper "Invexity and the Kuhn-Tucker Theorem"

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Abstract. Some remarks are made on the paper of Morgan A. Hanson "Invexity and the Kuhn-Tucker Theorem", appeared in *Journal of Mathematical Analysis* and its Applications (1999).

Keywords: invex functions, Kuhn-Tucker conditions, constraint qualifications.

M.A.Hanson[2] considered the following nonlinear programming problem

$$\begin{array}{c}\text{Minimize}_{x\in S}\ f(x)\\ \text{subject to } g(x)\le 0,\end{array} \qquad \text{((P))}$$

where $x\in\mathbb{R}^n$, $f\in\mathbb{R}$, $g\in\mathbb{R}^m$, and $f(x)$ and $g(x)$ are differentiable on an open set containing the feasible set *S*.

With regard to problem (P), $f(x)$ and $g(x)$ are *invex* at $u\in S$ with respect to a common function ("kernel function") $\eta(x,u)\in\mathbb{R}^n$, if for all $x\in S$

$$f(x)-f(u)\ge\eta(x,u)^T\nabla f(u)$$

and

$$g_i(x)-g_i(u)\ge\eta(x,u)^T\nabla g_i(u),\ i=1,\cdots,m.$$

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A slight generalization of the previous definition is *Type I invexity*, defined as follows, with reference to problem (P):

$$f(x) - f(u) \geq \eta(x, u)^T \nabla f(u)$$

and

$$-g_i(u) \geq \eta(x, u)^T \nabla g_i(u),\ i = 1, \cdots, m, \text{ for all } x \in S.$$

Then Hanson proves the following three theorems:

Theorem 1 *Suppose that problem (P) has a minimal solution at* $u \in S$. *If the Kuhn-Tucker conditions apply at u and all the Kuhn-Tucker multipliers* $y \in \mathbb{R}^m$ *are bounded, then the active constraint functions at u are invex with respect to a common* η *at u.*

Theorem 2 *Suppose that problem (P) has a minimal solution at* $u \in S$. *Suppose also that there is a point* $x \in S$ *such that* $g_i(x) < 0$ *for some* $i \in I = \{i : g_i(u) = 0\}$. *If the Kuhn-Tucker conditions apply at u and all the Kuhn-Tucker multipliers* $y \in \mathbb{R}^m$ *are bounded, then the active constraint functions and the objective function are Type I invex with respect to a common nontrivial* η *at u.*

Theorem 3 *Suppose that problem (P) has a minimal solution at* $u \in S$. *If the active constraint functions are invex with respect to a common* η *at u, then the Kuhn-Tucker conditions apply at u, provided that all the Kuhn-Tucker multipliers* $y \in \mathbb{R}^m$ *are bounded.*

The boundedness of $y \in \mathbb{R}^m$ is reasserted at the end of the paper, after a connection scheme. In the summary of [2] the author states that "invexity in itself constitutes an appropriate constraint qualification". On the previous results the following remarks have to be made.

If we denote by $\Lambda(u)$ the set of Kuhn-Tucker multipliers for problem (P) at $u \in S$, it is well known (see Gauvin [1]) that the following conditions are equivalent:

(i) The set $\Lambda(u)$ of Kuhn-Tucker multipliers is nonempty and bounded (more precisely: $\Lambda(u)$ is a compact convex polyhedron).

(ii) The Mangasarian-Fromovitz constraint qualification [3] holds at $u \in S$, i.e. there exists $z \in \mathbb{R}^n$ such that

$$\nabla g_i(u) z < 0,\ i \in I.$$

So, the requirement that all $y \in \Lambda(u)$ are bounded is equivalent to require the above constraint qualification (MFCQ) and thus it is not true that "constraint qualification requirements of the Kuhn-Tucker theory appear inherently through invexity" [2].

Other properties of the MFCQ (originally stated for a problem with both inequality and equality constraints) are the following ones:

a) The set of Fritz John multipliers $(y_0, y) \in \mathbb{R}^{m+1}$ for problem (P) where $y_0 = 0$, is empty if and only if the MFCQ holds at $u \in S$.

b) If the MFCQ holds at $u \in S$, then (see Still and Streng [4]) u is a strict local minimizer of order two (i.e. $f(x) \geq f(u) + \alpha \|x-u\|^2$ for all $x \in S \cap N(u,)$ with $\alpha > 0$) for problem (P) if and only if the usual second order sufficient conditions hold at u (here the functions involved in problem (P) are supposed to be twice-continuously differentiable):

$$z^T(\nabla_x^2 f(u) + \sum_{i \in I} y_i \nabla_x^2 g_i(u))z > 0,$$

for all $z \neq 0$, $z \in Z = \{z \in \mathbb{R}^n : \nabla f(u)z \leq 0, \nabla g_i(u)z \leq 0, i \in I\}$.

In this sense, there is no gap between second order necessary optimality conditions and second order sufficient optimality conditions.

On the grounds of Theorem 1 we can add the following property of the MFCQ:

c) If u is solution of problem (P) and MFCQ holds at u, then the active constraints are invex at u with respect to a common η (and obviously the Kuhn-Tucker conditions hold at u).

On the grounds of Theorem 2 we can add the following property of the MFCQ:

d) If u is a solution of problem (P), if there exists $x \in S$ such that $g_i(x) < 0$ for some $i \in I$ and the MFCQ holds at u, then f and every g_i $i \in I$, are Type I invex at u with respect to a common nontrivial η (and obviously the Kuhn-Tucker conditions hold at u).

Finally we note that Theorem 3 becomes trivial, as any constraint qualification assures that the Kuhn-Tucker conditions hold for a (local) solution of problem (P).

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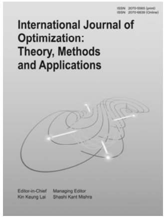

Duality for Nonlinear Programming under Generalized Kuhn-Tucker Condition

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Abstract. In this paper, KT-invex, weakly KT-pseudo-invex and type I problems with respect to different η_i are defined. In the framework of these concepts, a new Kuhn-Tucker type necessary condition is established, without using any constraint qualification, for nonlinear programming problems. In relation of the primal problem, Wolfe and Mond-Weir type dual programs are formulated involving a generalized Kuhn-Tucker condition. Weak, strong, converse and strict duality results are obtained for each dual program under the defined generalized invexity assumptions.

Keywords: nonlinear programming, weakly KT-pseudo-invex (type I) problem with respect to η and $(\theta_j)_j$, generalized Kuhn-Tucker condition, duality.

1 Introduction

Duality is one of the most important topics in optimization. In relation to a primal nonlinear programming, several dual programs are defined. Wolfe formulated in [23] one of the most known dual in the literature and established weak and strong duality results under the hypothesis that the functions occurring in the problem are convex.

One of the directions of research, following Wolfe's duality results, was to find ways to weaken the convexity requirements on the objective and constraint functions, while still maintaining the

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duality between the primal and Wolfe dual. Mangasarian [16], gives a set of examples to show that Wolfe duality results may not longer hold if the objective function is only pseudo-convex but not convex even if the constraints are linear. See also [5,15,21].

Hanson [9], introduced, for differentiable functions, the invexity notion with respect to η. He proved, under this generalized convexity, that Wolfe duality results are still hold. Subsequently, Hanson and Mond [11] introduced two new classes of functions which are not only sufficient, but are also necessary for optimality in primal and dual problems, respectively. They named these classes of functions type I and type II and obtained results concerning their optimality conditions, duality, and converse duality for a primal problem and the corresponding Wolfe dual. Further properties and applications of invexity for some general problems were studied in [1, 6, 7, 10, 12, 17, 18] and others. In [19], Mond and Weir, rather introduce a new form of generalized convex function, they modified the Wolfe dual by moving a part of the objective function to the constraints. They proved that duality holds between the primal and this new dual, called Mond-Weir dual, under the hypothesis of pseudo-invexity and quasi-invexity for the objective and constraint functions respectively. Note that for this dual, the objective functions of primal and dual are the same. In the literature, Wolfe and/or Mond-Weir (type) dual programs have been studied by Antczak [1,2,3], Stancu-Minasian [22], Kaul et al. [13], Kuk and Tanino [14], and others. Duality results are obtained using different concepts of generalized invex functions.

In this paper, we introduce new concepts of KT-invex, weak KT-pseudo-invex and type I problems in which each function occurring in these nonlinear programs is considered with respect to its own function η_i instead of a same function η. In the setting of these concepts, a new Kuhn-Tucker type necessary condition is established without using any constraint qualification. In relation to a nonlinear programming, and using the latter condition, two new dual programs in the format of Wolfe and Mond-Weir respectively are formulated, and different theorems of duality are proved for each dual program under the above generalized invexity assumptions.

2 Preliminaries and definitions

Invex functions were introduced to optimization theory by Hanson [9] (and called by Craven [8]) as a very broad generalization of convex functions.

Definition 1. [9] Let D be a nonempty open set of R^n, $f : D \to R$ be a differentiable function at $x_0 \in D$ and $\eta : D \times D \to R^n$. The function f is said to be:

- *invex* at x_0 with respect to η, if for each $x \in D$:

$$f(x) - f(x_0) \geq [\nabla f(x_0)]^t \eta(x, x_0).$$

- *pseudo-invex* at x_0 with respect to η, if for each $x \in D$:

$$[\nabla f(x_0)]^t \eta(x, x_0) \geq 0 \Rightarrow f(x) - f(x_0) \geq 0.$$

- *quasi-invex* at x_0 with respect to η, if for each $x \in D$:

$$f(x)-f(x_0)\le 0 \Rightarrow [\nabla f(x_0)]^t \eta(x,x_0)\le 0.$$

f is said to be invex (resp. pseudo-invex or quasi-invex) on *D* with respect to η, if *f* is invex (resp. pseudo-invex or quasi-invex) at each $x_0 \in D$ with respect to the same η.

Consider the following constrained nonlinear programming problem (P):

$$(P)\quad \begin{array}{c}\text{Minimize } f(x),\\ \text{subject to } g_j(x)\le 0,\ j=1,\cdots,k,\end{array}$$

where $f, g_j : D \to R, j=1,...,k$, *D* is an open set of R^n ; $X=\{x\in D: g_j(x)\le 0, j=1,...,k\}$ is the set of feasible solutions for (P). For $x_0 \in D$, we denote $J(x_0)=\{j\in\{1,...,k\}: g_j(x_0)=0\}$.

The problem (P) is said to be HC-invex at $x_0 \in X$ if f and $g_j, j=1,...,k$ are invex at x_0 (with respect to the same function η). Thus, if the problem (P) is HC-invex, then every Kuhn-Tucker point is a minimizer of (P) [9]. Martin [17] remarked that the converse is not true in general, and he proposed a weaker notion, called KT-invexity, which assures that every Kuhn-Tucker point is a minimizer of problem (P) if and only if problem (P) is KT-invex.

Definition 2. [17] The problem (P) is said to be *KT-invex* on the feasible set *X* with respect to η, if the functions *f* and g are differentiable on *X* and there exists $\eta: X\times X \to R^n$ such that for each $x, x_0 \in X$:

$$f(x)-f(x_0)\ge [\nabla f(x_0)]^t \eta(x,x_0),$$
$$-[\nabla g_j(x_0)]^t \eta(x,x_0)\ge 0, \forall j\in J(x_0).$$

However, there exists no always a function η for which the problem (P) is KT-invex. The following example illustrate this fact.

Example 1. In the problem (P), if we have $D=]-1,1[^2, f(x=(x_1,x_2))=x_1^3+x_2, g_1(x)=-x_1+x_2, g_2(x)=log(1+x_1)-x_2, g_3(x)=x_1+x_2^2, g_4(x)=10x_1^3-x_2,$ there exists no a function $\eta: X\times X\to R^2$ for which the problem (P) is KT-invex at the feasible point $x_0=(0,0)$. In fact, the only $\eta: X\times X\to R^2$ such that $-[\nabla g_j(x_0)]^t\eta(x,x_0)\ge 0, \forall x\in X, \forall j=1,\cdots,4,$ is $\eta\equiv 0$ and *f* is not invex at x_0 with respect to $\eta\equiv 0$ (take $x=\left(\frac{-1}{2},\frac{-2}{3}\right)$).

Now, we introduce a new classes of KT-invex and weakly KT-pseudo-invex problems by taking each function occurring in the problem (P) with respect to its own function η or θ_j .

Definition 3. The problem (P) is said to be *KT-invex* at $x_0 \in D$ with respect to η and $(\theta_j)_{j\in J(x_0)}$, if the functions *f* and *g* are differentiable at x_0 and there exist $\eta: X\times D \to R^n$ and $\theta_j : X\times D \to R^n, j\in J(x_0)$ such that for each $x\in X$:

$$f(x)-f(x_0)\ge [\nabla f(x_0)]^t \eta(x,x_0),$$
$$-[\nabla g_j(x_0)]^t \theta_j(x,x_0)\ge 0, \forall j\in J(x_0).$$

The problem (P) is said to be KT-invex on D with respect to η and $(\theta_j)_j$, if it is KT-invex at each $x_0 \in D$ with respect to the same η and $(\theta_j)_{j\in J(x_0)}$.

Definition 4. The problem (P) is said to be *weakly KT-pseudo-invex* at $x_0 \in D$ with respect to η and $(\theta_j)_{j\in J(x_0)}$, if the functions f and g are differentiable at x_0 and there exist $\eta : X\times D \to R^n$ and $\theta_j : X\times D \to R^n$, $j\in J(x_0)$ such that for each $x\in X$:

$$f(x)-f(x_0)<0 \Rightarrow \exists\, \overline{x}\in X, \begin{cases} [\nabla f(x_0)]^t \eta(\overline{x},x_0)<0, \\ \left[\nabla g_j(x_0)\right]^t \theta_j(\overline{x},x_0)\le 0, \forall j\in J(x_0). \end{cases} \tag{1}$$

If $\overline{x}=x$, in the relation (1), we say that (P) is KT-pseudo-invex at x_0 with respect to η and $(\theta_j)_{j\in J(x_0)}$. The problem (P) is said to be (weakly) KT-pseudo-invex on D with respect to η and $(\theta_j)_j$, if it is (weakly) KT-pseudo-invex at each $x_0 \in D$ with respect to the same η and $(\theta_j)_{j\in J(x_0)}$. In the relation (1), if we have $f(x)-f(x_0)\le 0$ ($x\ne x_0$), instead of $f(x)-f(x_0)<0$, we say that (P) is semi strictly (weakly) KT-pseudo-invex at x_0 with respect to η and $(\theta_j)_{j\in J(x_0)}$.

If (P) is KT-pseudo-invex at x_0 with respect to η and $(\theta_j)_{j\in J(x_0)}$, then it is weakly KT-pseudo-invex at x_0 with respect to the same η and $(\theta_j)_{j\in J(x_0)}$ (with $\overline{x}=x$) but the converse is not true.

Example 2. Consider the problem (P) with $f(x)=-x_1^2-x_2$ and $g(x)=x_2$ where $f,g: R^2\to R$. The problem is weakly KT-pseudo-invex at the feasible point $x_0=(1,0)$ with respect to $\eta(x,x_0)=(x_0-x)\in R^2$ and $\theta(x,x_0)=(x-x_0)\in R^2$ (take $\overline{x}=[(f(x)-f(x_0), f(x)-f(x_0))+x_0]\in R^2$). But it is not KT-pseudo-invex at x_0 with respect to the same η and θ (take $x=(2,-1)$).

In the following example, we give a problem such that there exists no function η for which it is KT-invex at a feasible point. However, we show that it is KT-invex (KT-pseudo-invex) with respect to different η and $(\theta_j)_j$.

Example 3. Reconsider the problem given in the example 1. The problem is KT-invex at x_0 with respect to $\eta(x,x_0)=(x_1^2-x_2, x_1^3+x_2)$, $\theta_1(x,x_0)=(x_1^2,-x_1^2), \theta_2(x,x_0)=(-x_2^2,x_2^2), \theta_3(x,x_0)=(-x_1^2,0)$ and $\theta_4(x,x_0)=(0,x_2^2)$. In the case of $f(x)=e^{x_2}-x_1$, there exists no a function $\eta: X\times X\to R^2$ with $\eta\ne 0$ for which the problem (P) is KT-invex at x_0. But the same problem is KT-pseudo-invex at x_0 with respect to $\eta(x,x_0)=(1+x_1-e^{x_2}, e^{x_2}-x_1-1)$ and $(\theta_j)_{j=\overline{1,4}}$ given above.

Next, with a slight modification on the constraint inequalities, in the concept of KT-invex problem (definition 3), we define the concept of type I problem with respect to different functions η and $(\theta_j)_{j=\overline{1,k}}$, given as follows.

Definition 5. The problem (P) is said to be of *type I* at $x_0\in D$ with respect to η and $(\theta_j)_{j=\overline{1,k}}$, if the functions f and g are differentiable at x_0 and there exist $\eta: X\times D\to R^n$ and $\theta_j : X\times D\to R^n$, $j=\overline{1,k}$ such that for each $x\in X$:

$$f(x)-f(x_0)\ge [\nabla f(x_0)]^t \eta(x,x_0), \tag{2}$$

$$-g_j(x_0) \geq [\nabla g_j(x_0)]^t \theta_j(x, x_0), \forall j = 1, \cdots, k. \quad (3)$$

The problem (P) is said to be of type I on D with respect to η and $(\theta_j)_j$, if it is of type I at each $x_0 \in D$ with respect to the same η and $(\theta_j)_{j=\overline{1,k}}$. If a strict inequality holds in (2) (whenever $x \neq x_0$), we say that (P) is semi strictly type I at x_0 with respect to η and $(\theta_j)_{j=\overline{1,k}}$.

Remark 1. In the case of $\theta_j = \eta, \forall j = \overline{1,k}$, the concept of type I problem given in the definition 5 reduce to the one of type I functions given in [11,13].

The relationship between these class of problems is as follows.

Property 1. Let $x_0 \in D$.

- If (P) is type I at x_0 with respect to η and $(\theta_j)_{j=\overline{1,k}}$ then (P) is KT-invex at x_0 with respect to η and $(\theta_j)_{j\in J(x_0)}$ and then (P) is (weakly) KT-pseudo-invex at x_0 with respect to the same η and $(\theta_j)_{j\in J(x_0)}$.
- (P) is KT-invex at x_0 with respect to η and $(\theta_j)_{j\in J(x_0)}$, if and only if (P) is of type I at x_0 with respect to the same η and $(\theta_j)_{j\in J(x_0)}$.

In the setting of the new concepts of KT-invex and type I problems with respect to different η and $(\theta_j)_j$, introduced above, we establish a new Kuhn-Tucker type necessary condition for the problem (P).

Theorem 1. (Kuhn-Tucker type necessary optimality condition) Suppose that x_0 is a local or global solution for (P) and the functions f, $g_j, j \in J(x_0)$ are differentiable at x_0. Then there exist vector functions $\eta : X \times D \to R^n, \theta_j : X \times D \to R^n, j \in J(x_0)$ and $\lambda \in R_+^{|J(x_0)|}$ with $\lambda_j > 0, j \in J(x_0)$ such that $(x_0, \lambda, \eta, (\theta_j)_j)$ satisfies the following generalized Kuhn-Tucker condition.

$$[\nabla f(x_0)]^t \eta(x, x_0) + \sum_{j\in J(x_0)} \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \geq 0, \forall x \in X. \quad (4)$$

Proof: Suppose that x_0 is a local solution of (P). Then there exists, a neighborhood of x_0, $v(x_0) \subset X$ such that for all $x \in v(x_0)$, $f(x) - f(x_0) \geq 0$. Thus, it suffices to take $\eta, \theta_j, j \in J(x_0)$ and λ as follows:

- $\eta(x, x_0) = \begin{cases} [f(x) - f(x_0)]t(x_0), & \text{if } x \in v(x_0), \\ t(x_0), & \text{otherwise,} \end{cases} \quad , t(x_0) \in R^n$

  with $t_i(x_0) = \begin{cases} 1, & \text{if } \frac{\partial f}{\partial x_i}(x_0) \geq 0, \\ -1, & \text{otherwise,} \end{cases}$ for all $i = 1, \cdots, n$;

- $\theta_j(x, x_0) = -g_j(x)s^j(x_0), x \in X, s^j(x_0) \in R^n$

  with $s_i^j(x_0) = \begin{cases} 1, & \text{if } \frac{\partial g_j}{\partial x_i}(x_0) \geq 0, \\ -1, & \text{otherwise,} \end{cases}$ for all $i = 1, \cdots, n$;

- $\lambda_j = \dfrac{1}{|J(x_0)|}$ for all $j \in J(x_0)$.

If x_0 is a global solution of (P), we take $\eta(x, x_0) = [f(x) - f(x_0)]t(x_0), \forall x \in X$ instead of the one given above.

Remark 2. Note that the necessary generalized Kuhn-Tucker condition is obtained without requiring any constraint qualification contrary to that usual one given by Kuhn-Tucker.

Remark 3. In the proof of theorem 1, we have not used (any) alternative theorem to prove the necessary condition, unlike the usual procedure. Furthermore, the alternative theorems, when they are used, they give us only the existence of the kernel η and/or the Lagrange multipliers λ_j (for example) but in our case, we prove the existence of η, $(\theta_j)_j$ and the Lagrange multipliers λ_j and we construct them explicitly.

3 Wolfe Type Duality

In relation to (P) and using the generalized Kuhn-Tucker condition (4), we formulate the following dual problem which is in the format of Wolfe [23].

$$(WD) \quad \text{Maximize } f(y) + \lambda^t g(y),$$

Subject to

$$[\nabla f(y)]^t \eta(x, y) + \sum_{j=1}^{k} \lambda_j [\nabla g_j(y)]^t \theta_j(x, y) \geq 0, \forall x \in X,$$
$$y \in D, \lambda \in R_+^k, \eta : X \times D \to R^n,$$
$$\theta_j : X \times D \to R^n, \forall j = 1, \cdots, k.$$

Let $Y = \Big\{(y, \lambda, \eta, (\theta_j)_{j=\overline{1,k}}) : [\nabla f(y)]^t \eta(x, y) + \sum_{j=1}^{k} \lambda_j [\nabla g_j(y)]^t \theta_j(x, y) \geq 0, \forall x \in X;$

$y \in D, \lambda \in R_+^k, \eta : X \times D \to R^n, \theta_j : X \times D \to R^n, \forall j = 1, ..., k\Big\}$ the set of feasible solutions of problem (WD). We denote by $Pr_D Y$ the projection of the set Y on D, that is, by a definition $Pr_D Y = \{y \in D : (y, \lambda, \eta, (\theta_j)_{j=\overline{1,k}}) \in Y\}$.

It is known that weak and strong duality in the sense of Wolfe hold in the case when the functions occurring in the Wolfe dual problem, are convex [4,16] or (generalized) invex with respect to the same function η [1,2,3,9,20]. Now, we establish certain duality results between (P) and (WD) by using the concepts of type I and weakly KT-pseudo-invex problems with respect to different η and $(\theta_j)_j$.

Theorem 2. (Weak duality) Suppose that

(i) $x \in X$;

(ii) $(y,\lambda,\eta,(\theta_j)_{j=\overline{1,k}})\in Y$;
(iii) the problem (P) is type I at y with respect to η and $(\theta_j)_{j=\overline{1,k}}$.

Then $f(x)\nless f(y)+\lambda^t g(y)$.

Proof: From the hypothesis (iii), we obtain

$$-g_j(y)\ge[\nabla g_j(y)]^t\theta_j(x,y),\forall j=1,\cdots,k,$$

by $\lambda_j\ge 0,\forall j=1,\cdots,k$, it follows that

$$-\sum_{j=1}^{k}\lambda_j g_j(y)\ge\sum_{j=1}^{k}\lambda_j[\nabla g_j(y)]^t\theta_j(x,y). \tag{5}$$

Now, the inequality (5) and the hypothesis (*ii*) and (*iii*) give

$$f(x)-[f(y)+\lambda^t g(y)]\ge[\nabla f(y)]^t\eta(x,y)+\sum_{j=1}^{k}\lambda_j[\nabla g_j(y)]^t\theta_j(x,y)\ge 0.$$

Hence $f(x)\nless f(y)+\lambda^t g(y)$.

Now, we establish the following strong duality result between (P) and (WD) without using any constraint qualification.

Theorem 3. (Strong duality) Suppose that
(i) x_0 is an optimal solution for (P);
(ii) the functions f, $g_j, j\in J(x_0)$ are differentiable at x_0.

Then there exist vector functions $\eta: X\times D\to R^n,\theta_j: X\times D\to R^n, j=\overline{1,k}$ and $\lambda\in R_+^k$ such that $(x_0,\lambda,\eta,(\theta_j)_{j=\overline{1,k}})\in Y$ and the objective functions of (P) and (WD) have the same values at x_0 and $(x_0,\lambda,\eta,(\theta_j)_{j=\overline{1,k}})$, respectively. If, further, the problem (P) is type I at any $\bar{y}\in Pr_D Y$ with respect to $\bar{\eta}$ and $(\bar{\theta}_j)_{j=\overline{1,k}}$ (with $(\bar{y},\bar{\lambda},\bar{\eta},(\bar{\theta}_j)_{j=\overline{1,k}})\in Y$), then $(x_0,\lambda,\eta,(\theta_j)_{j=\overline{1,k}})\in Y$ is an optimal solution of (WD).

Proof: Let $K=\{1,\cdots,k\}$. By the theorem 1, there exist vector functions $\eta: X\times D\to R^n,\theta_j$: $X\times D\to R^n, j\in J(x_0)$ and $\lambda\in R_+^{|J(x_0)|}$ with $\lambda_j>0, j\in J(x_0)$ such that

$$[\nabla f(x_0)]^t\eta(x,x_0)+\sum_{j\in J(x_0)}\lambda_j[\nabla g_j(x_0)]^t\theta_j(x,x_0)\ge 0,\forall x\in X.$$

It follows that, by setting for all $j\in K-J(x_0),\lambda_j=0$ and $\theta_j(x,x_0)$ any function, $(x_0,\lambda,\eta,(\theta_j)_{j=\overline{1,k}})\in Y$. The objective function values of (P) and (WD) are equal because of the choice of the scalars $\lambda_j, j=\overline{1,k}$.

Next, suppose that $(x_0,\lambda,\eta,(\theta_j)_{j=\overline{1,k}})\in Y$ is not an optimal solution of (WD). Then there exists $(y^*,\lambda^*,\eta^*,(\theta_j^*)_{j=\overline{1,k}})\in Y$ such that $f(x_0)<f(y^*)+\lambda^{*t}g(y^*)$, which violates the weak duality theorem 2. Hence $(x_0,\lambda,\eta,(\theta_j)_{j=\overline{1,k}})\in Y$ is indeed an optimal solution of (WD).

Then, we state the converse duality theorem to the problems (P) and (WD).

Theorem 4. (Converse duality) Suppose that

(i) $(y,\lambda,\eta,(\theta_j)_{j=\overline{1,k}})\in Y$;

(ii) $y\in X$;

(iii) the problem (P) is weakly KT-pseudo-invex at y with respect to η and $(\theta_j)_{j=\overline{1,k}}$.

Then y is an optimal solution for (P).

Proof: Let us suppose that y is not an optimal solution of (P). Then there exists a feasible point x such that $f(x)-f(y)<0$.

Since (P) is weakly KT-pseudo-invex at y with respect to η and $(\theta_j)_{j=\overline{1,k}}$, it follows that

$$\exists\, \overline{x}\in X, \begin{cases} [\nabla f(y)]^t\eta(\overline{x},y)<0, \\ \left[\nabla g_j(y)\right]^t\theta_j(\overline{x},y)\le 0, \forall j\in\{1,\cdots,k\} \end{cases} \tag{6}$$

As $\lambda\ge 0$ and from (6), we obtain

$$[\nabla f(y)]^t\eta(\overline{x},y)+\sum_{j=1}^{k}\lambda_j[\nabla g_j(y)]^t\theta_j(\overline{x},y)<0,$$

which contradicts (*i*), and therefore, y is an optimal solution of (P).

Next, we set up a strict duality result given as follows.

Theorem 5. (Strict duality) Suppose that

(i) $x\in X$ and $(y,\lambda,\eta,(\theta_j)_{j=\overline{1,k}})\in Y$ such that

$$f(x)=f(y)+\lambda^t g(y); \tag{7}$$

(ii) the problem (P) is semi strictly type I at y with respect to η and $(\theta_j)_{j=\overline{1,k}}$.

Then $x=y$.

Proof: We proceed by contradiction. Suppose that $x\neq y$.

From the hypothesis (*ii*), we obtain

$$-g_j(y)\ge[\nabla g_j(y)]^t\theta_j(x,y), \forall j=1,\cdots,k,$$

by $\lambda_j\ge 0, \forall j=1,\cdots,k$ and (*i*), it follows that

$$-\sum_{j=1}^{k}\lambda_j g_j(y)\ge\sum_{j=1}^{k}\lambda_j[\nabla g_j(y)]^t\theta_j(x,y)\ge-[\nabla f(y)]^t\eta(x,y). \tag{8}$$

Since f is strictly invex at y with respect to η (hypothesis (ii)), we have

$$f(x)-f(y)>[\nabla f(y)]^t\eta(x,y),$$

which on using (8) gives a contradiction to the given hypothesis (7). Hence $x=y$.

4 Mond-Weir Type Duality

In relation to (P) and using the generalized Kuhn-Tucker condition (4), we formulate the following dual problem which is in the format of Mond-Weir [19].

$$(MWD) \quad \text{Maximize } f(y),$$

Subject to

$$[\nabla f(y)]^t \eta(x,y) + \sum_{j\in J(y)} \lambda_j [\nabla g_j(y)]^t \theta_j(x,y) \geq 0, \forall x \in X,$$
$$y \in D, \lambda \in R_+^{|J(y)|}, \eta : X \times D \to R^n,$$
$$\theta_j : X \times D \to R^n, \forall j \in J(y).$$

Let $\overline{Y} = \Big\{ (y,\lambda,\eta,(\theta_j)_{j\in J(y)}) : [\nabla f(y)]^t \eta(x,y) + \sum_{j\in J(y)} \lambda_j [\nabla g_j(y)]^t \theta_j(x,y) \geq 0, \forall x \in X;$

$y \in D, \lambda \in R_+^{|J(y)|}; \eta : X \times D \to R^n, \theta_j : X \times D \to R^n, \forall j \in J(y) \Big\}$ the set of feasible solutions of problem (MWD). We denote by $Pr_D\overline{Y}$ the projection of the set $\overline{Y}$ on D, that is, by a definition $Pr_D\overline{Y} = \{ y \in D : (y,\lambda,\eta,(\theta_j)_{j\in J(y)}) \in \overline{Y} \}$.

In what follows, we establish some duality results between (P) and (MWD) by using the concept of weak KT-pseudo-invexity with respect to different η and $(\theta_j)_j$.

Theorem 6. (Weak duality) Suppose that

(i) $x \in X$;

(ii) $(y,\lambda,\eta,(\theta_j)_{j\in J(y)}) \in \overline{Y}$;

(iii) the problem (P) is weakly KT-pseudo-invex at y with respect to η and $(\theta_j)_{j\in J(y)}$.

Then $f(x) \nless f(y)$.

Proof: We proceed by contradiction. Suppose that $f(x) < f(y)$.

Since (P) is weakly KT-pseudo-invex at y with respect to η and $(\theta_j)_{j\in J(y)}$, it follows that

$$\exists\, \overline{x} \in X, \begin{cases} [\nabla f(y)]^t \eta(\overline{x},y) < 0, \\ [\nabla g_j(y)]^t \theta_j(\overline{x},y) \leq 0, \forall j \in J(y). \end{cases} \tag{9}$$

As $\lambda \geq 0$ and from (9), we obtain

$$[\nabla f(y)]^t \eta(\overline{x},y) + \sum_{j\in J(y)} \lambda_j [\nabla g_j(y)]^t \theta_j(\overline{x},y) < 0,$$

which contradicts (ii), and the conclusion follows.

Now, we establish the following strong duality result between (P) and (MWD) without using any constraint qualification.

Theorem 7. (Strong duality) Suppose that
(i) x_0 is an optimal solution for (P);
(ii) the functions f, $g_j, j \in J(x_0)$ are differentiable at x_0.

Then there exist vector functions $\eta : X \times D \to R^n, \theta_j : X \times D \to R^n, j \in J(x_0)$ and $\lambda \in R_+^{|J(x_0)|}$ with $\lambda_j > 0, j \in J(x_0)$ such that $(x_0, \lambda, \eta, (\theta_j)_{j \in J(x_0)}) \in \overline{Y}$ and the objective functions of (P) and (MWD) have the same values at x_0 and $(x_0, \lambda, \eta, (\theta_j)_{j \in J(x_0)})$, respectively. If, further, the problem (P) is weakly KT-pseudo-invex at any $\overline{y} \in Pr_D \overline{Y}$ with respect to $\overline{\eta}$ and $(\overline{\theta}_j)_{j \in J(\overline{y})}$ (with $(\overline{y}, \overline{\lambda}, \overline{\eta}, (\overline{\theta}_j)_{j \in J(\overline{y})}) \in \overline{Y}$), then $(x_0, \lambda, \eta, (\theta_j)_{j \in J(x_0)}) \in \overline{Y}$ is an optimal solution of (MWD).

Proof: By the theorem 1, there exist vector functions $\eta : X \times D \to R^n, \theta_j : X \times D \to R^n, j \in J(x_0)$ and $\lambda \in R_+^{|J(x_0)|}$ with $\lambda_j > 0, j \in J(x_0)$ such that

$$[\nabla f(x_0)]^t \eta(x, x_0) + \sum_{j \in J(x_0)} \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \geq 0, \forall x \in X.$$

It follows that $(x_0, \lambda, \eta, (\theta_j)_{j \in J(x_0)}) \in \overline{Y}$.

Trivially, the objective function values of (P) and (MWD) are equal.

Next, suppose that $(x_0, \lambda, \eta, (\theta_j)_{j \in J(x_0)}) \in \overline{Y}$ is not an optimal solution of (MWD). Then there exists $(y^*, \lambda^*, \eta^*, (\theta_j^*)_{j \in J(y^*)}) \in \overline{Y}$ such that $f(x_0) < f(y^*)$, which violates the weak duality theorem 6. Hence $(x_0, \lambda, \eta, (\theta_j)_{j \in J(x_0)}) \in \overline{Y}$ is indeed an optimal solution of (MWD).

Then, we state the converse duality theorem to the problems (P) and (MWD).

Theorem 8. (Converse duality) Suppose that
(i) $(y, \lambda, \eta, (\theta_j)_{j \in J(y)}) \in \overline{Y}$;
(ii) $y \in X$;
(iii) the problem (P) is weakly KT-pseudo-invex at y with respect to η and $(\theta_j)_{j \in J(y)}$.

Then y is an optimal solution for (P).

Proof: The proof of this theorem is similar to that theorem 4.

Next, we set up a strict duality result given as follows.

Theorem 9. (Strict duality) Suppose that
(i) $x \in X$ and $(y, \lambda, \eta, (\theta_j)_{j \in J(y)}) \in \overline{Y}$ such that $f(x) = f(y)$;
(ii) the problem (P) is semi strictly weakly KT-pseudo-invex at y with respect to η and $(\theta_j)_{j \in J(y)}$.

Then $x = y$.

Proof: We proceed by contradiction. Suppose that $x \neq y$.

By hypothesis we have $f(x) = f(y)$ and since (P) is semi strictly weakly KT-pseudo-invex at y with respect to η and $(\theta_j)_{j \in J(y)}$, it follows that

$$\exists\, \overline{x} \in X, \begin{cases} [\nabla f(y)]^t \eta(\overline{x}, y) < 0, \\ [\nabla g_j(y)]^t \theta_j(\overline{x}, y) \leq 0, \forall j \in J(y). \end{cases} \tag{10}$$

As $\lambda \geq 0$ and from (10), we obtain

$$[\nabla f(y)]^t \eta(\bar{x}, y) + \sum_{j \in J(y)} \lambda_j [\nabla g_j(y)]^t \theta_j(\bar{x}, y) < 0,$$

which contradicts the fact that $(y, \lambda, \eta, (\theta_j)_{j \in J(y)}) \in \bar{Y}$. Hence $x = y$.

5 Conclusion

In this paper, we have defined new classes of problem called KT-invex, (weakly) KT-pseudo-invex and type I with respect to different functions η and $(\theta_j)_j$ as a generalization of KT-invex and type I problems with respect to the same function η. In the setting of this new concepts of problems, we have established a new Kuhn-Tucker type necessary condition, without using any constraint qualification and any alternative theorem, for nonlinear programming problems. Two new dual programs in the format of Wolfe and Mond-Weir respectively are formulated involving a generalized Kuhn-Tucker condition. For each dual program, weak, strong, converse and strict duality results are obtained under the defined generalized invexity.

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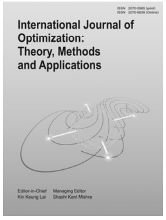

Strong Convergence Theorem for Generalized Equilibrium Problems and Countable Family of Nonexpansive Mappings

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Abstract. We introduce an iterative method for finding the common element of the set of common fixed points of nonexpansive mappings, the set of solutions of a generalized equilibrium problem, and the set of solutions of the variational inequality. We show that the sequence converges strongly to a common element of the above three sets. Moreover, we apply our result to the problem of finding a common fixed point of a countable family of nonexpansive mappings, and the problem of finding a zero of a monotone operator.

Keywords: nonexpansive mapping, inverse-strongly monotone mapping, generalized equilibrium, variational inequality.

1 Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. Let C be a nonempty closed convex subset of H, let $F: C\times C \to R$ be a bifunction and let $A: C \to H$ be a nonlinear mapping. The generalized equilibrium problem for F is to find $z \in C$ such that

$$F(z, y) + \langle Az, y - z \rangle \geq 0, \forall y \in C. \tag{1.1}$$

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The set of such $z \in C$ is denoted by EP, i.e.,

$$EP = \{z \in C : F(z, y) + \langle Az, y - z \rangle \geq 0, \forall y \in C\}.$$

In the case of $A \equiv 0, EP$ is denoted by $EP(F)$. In the case of $F \equiv 0, EP$ is also denoted by $VI(A,C)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others. A mapping S of C into itself is nonexpansive if

$$\| Sx - Sy \| \leq \| x - y \|, \forall x, y \in C.$$

The set of fixed point of S is defined by $F(S)$. A mapping A of C into H is called α-inverse-strongly monotone [1,2] if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \| Au - Av \|^2$$

for all $u, v \in C$. It is obvious that any α-inverse-strongly monotone mapping A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. Very recently, Moudafi [3] introduced an iterative method for finding an element of $EP \bigcap F(S)$, where $A : C \to H$ is an inverse-strongly monotone mapping and then proved a weak convergence theorem.

On the other hand, Yao et al. [4] introduced the following iterative scheme for finding an element of $F(S) \bigcap VI(A, C)$. Let C be a closed convex subset of a real Hilbert space H, $A : C \to H$ a monotone, L-Lipschitz continuous mapping, and S a nonexpansive mapping of C into itself such that $F(S) \bigcap VI(A, C) \neq \emptyset$. Suppose that $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by

$$y_n = P_C(x_n - \lambda_n A_{x_n})$$

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n),$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ and $\{\lambda_n\} \subseteq (0,1)$. They proved that the sequence $\{x_n\}$ defined by above converges strongly to a common element of $F(S) \bigcap VI(A, C)$.

In this paper, motivated by Moudafi [3] and Yao et al. [4], we introduce another iterative method for finding an element of $\bigcap_{n=1}^{\infty} F(S_n) \bigcap EP \bigcap VI(A, C)$, where $A : C \to H$ is an inverse-strongly monotone mapping and obtain a strong convergence theorem.

2 Preliminaries

Let H be a real Hilbert space, C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\| x - P_C x \| \leq \| x - y \|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C. We know that P_C is a firmly nonexpansive mapping from H onto C, i.e.,

$$\| P_C x - P_C y \|^2 \leq \langle P_C x - P_C y, x - y \rangle, \forall x, y \in H.$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x-P_C x, y-P_C x\rangle \le 0, \tag{2.1}$$

$$\| x-y\|^2 \ge \| x-P_C x\|^2 + \| y-P_C x\|^2 \tag{2.2}$$

for all $x\in H, y\in C$. It is easy to see that the following is true:

$$u\in VI(A,C) \Longleftrightarrow u=P_C(u-\lambda Au), \lambda>0.$$

A mapping $T:H\to 2^H$ is called monotone if for all $x,y\in H$, $f\in Tx, g\in Ty$ imply $\langle x-y, f\,.\,-g\rangle \ge 0$ A monotone T is maximal if the graph of T is not properly contained in the graph of any other monotone mapping. Let B be a monotone mapping of C into H, L-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v\in C$, that is, $N_C v=\{\omega\in H:\langle v-u,\omega\rangle\ge 0, \forall u\in C\}$. Define

$$Tv=\{Bv+N_C v, v\in C, \emptyset, v\notin C.$$

Then T is the maximal monotone and $T^{-1}(0)=VI(B,C)$, see [5].

We need the following Lemmas for the proof of our main results.

Lemma 2.1 (see [6]). *Let E be an inner product space. Then for all $x,y,z\in E$ and $\alpha,\beta,\gamma\in[0,1]$ with $\alpha+\beta+\gamma=1$,*

$$\| \alpha x+\beta y+\gamma z\|^2=\alpha\| x\|^2+\beta\| y\|^2+\gamma\| z\|^2-\alpha\beta\| x-y\|^2-\alpha\gamma\| x-z\|^2-\beta\gamma\| y-z\|^2.$$

Lemma 2.2 (see [7]). *Let $\{x_n\},\{z_n\}$ be bounded sequence in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] with $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$. Suppose $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n, n\ge 1$ and $\limsup_{n\to\infty}(\| z_{n+1}-z_n\|-\| x_{n+1}-x_n\|)\le 0$. Then, $\lim_{n\to\infty}\| z_n-x_n\|=0$.*

Lemma 2.3 (see [8]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1}\le(1-\alpha_n)a_n+\delta_n, n\ge 1,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=1}^{\infty}\alpha_n=\infty$ *and*
(ii) $\limsup_{n\to\infty}(\frac{\delta_n}{\alpha_n})\le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n|<\infty$.

Then $\lim_{n\to\infty}a_n=0$.

Lemma 2.4 (see [9]). *Let C be a nonempty closed subset of a Banach space and $\{S_n\}$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty}\sup\{\| S_{n+1}z-S_n z\|: z\in C\}<\infty$. Then, for each $y\in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy=\lim_{n\to\infty}S_n y$, $\forall y\in C$. Then $\lim_{n\to\infty}\sup\{\| Sz-S_n z\|: z\in C\}=0$.*

For solving the generalized equilibrium problem for a bifunction $F:C\times C\to R$, let us assume that F satisfies the following conditions:

(A1) $F(x,x)=0$, $\forall x \in C$;
(A2) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0$, $\forall x,y \in C$;
(A3) $\lim_{t\to 0} F(tz+(1-t)x,y)\leq F(x,y)$, $\forall x,y,z \in C$;
(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

We know the following lemma; see, for instance, [10] and [11].

Lemma 2.5. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C\times C$ into R satisfying (A1)-(A4). Then, for any $r>0$ and $x\in H$, there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle \geq 0, \forall y\in C.$$

Further, if $T_r x=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\forall y\in C\}$, then the following hold:

(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e.,

$$\| T_r x-T_r y\|^2\leq \langle T_r x-T_y, x-y\rangle, \forall x,y\in H;$$

(3) $F(T_r)=EP(F)$;
(4) $EP(F)$ is closed and convex.

3 Main Results

In this section, we prove a strong convergence theorem which is our main results.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ into R satisfying (A1)-(A4), $A: C\to H$ an α-inverse-strongly monotone mapping and let S_n ba a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n)\bigcap VI(A,C)$ $\bigcap EP\neq \emptyset$. Let the sequence $\{x_n\},\{u_n\},\{y_n\}$ be generated by*

$$\begin{aligned} &x_1=x\in C \quad \text{chosen arbitrary},\\ &F(u_n,y)+\langle Ax_n, y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\forall y\in C,\\ &y_n=P_C(u_n-\lambda_n Au_n),\\ &x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n S_n P_C(u_n-\lambda_n Ay_n), n\geq 1, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subseteq[0,1]$, $\{\lambda_n\}\subset(0,1)$, and $\{r_n\}\subset(0,2\alpha]$ satisfy the following conditions:

(C1) $\alpha_n+\beta_n+\gamma_n=1$,
(C2) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}=\infty$,
(C3) $0<\inf\sup_{n\to\infty}\beta_n\leq\lim\sup_{n\to\infty}\beta_n<1$,
(C4) $\lim_{n\to\infty}\lambda_n=0$,
(C5) $\lim\inf_{n\to\infty} r_n>0$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$.

Suppose that $\sum_{n=1}^{\infty}\sup\|S_{n+1}-S_n z\|: z\in B<\infty$ *for any bounded subset B of C. Let S be a mapping of C into itself defined by* $Sy=\lim_{n\to\infty}S_n y$ *for all* $y\in C$ *and suppose that* $F(S)=\bigcap_{n=1}^{\infty}F(S_n)$. *Then the sequences* $\{x_n\},\{y_n\},\{u_n\}$ *converge strongly to the same point q, where* $q=P_{\bigcap_{n=1}^{\infty}F(S_n)\bigcap VI(A,C)\bigcap EP}f(q)$.

Proof: Let $Q=P_{\cap_{n=1}^{\infty}F(S_n)\cap VI(A,C)\cap EP}$. Since *f* is a contraction, *Qf* is a contraction of *C* into itself. Then *Qf* has a unique fixed point $q\in C$. We divide the proof into several steps.

Step 1 ($\{x_n\}$ is bounded). Let $t_n=P_C(u_n-\lambda_n Ay_n)$ for all $n\geq 1$ and $x^*\in\bigcap_{n=1}^{\infty}F(S_n)\bigcap VI(A,C)\bigcap EP$. It follows from (2.2) that

$$\begin{aligned}\|t_n-x^*\|^2&\leq\|u_n-\lambda_n Ay_n-x^*\|^2-\|u_n-\lambda_n Ay_n-t_n\|^2\\&=\|u_n-x^*\|^2-2\lambda_n\langle Ay_n,u_n-x^*\rangle+\lambda_n^2\|Ay_n\|^2\\&\quad-\|u_n-t_n\|^2+2\lambda_n\langle Ay_n,u_n-t_n\rangle-\lambda_n^2\|Ay_n\|^2\\&=\|u_n-x^*\|^2+2\lambda_n\langle Ay_n,x^*-t_n\rangle-\|u_n-t_n\|^2\\&=\|u_n-x^*\|^2-\|u_n-t_n\|^2+2\lambda_n\langle Ay_n-Ax^*,x^*-y_n\rangle\\&\quad+2\lambda_n\langle Ax^*,x^*-y_n\rangle+2\lambda_n\langle Ay_n,y_n-t_n\rangle.\end{aligned}\tag{3.2}$$

Since *A* is monotone and $x^*\in VI(A,C)$, we have

$$\langle Ay_n-Ax^*,x^*-y_n\rangle\leq 0,\quad\langle Ax^*,x^*-y_n\rangle\leq 0.$$

This together with (3.2) implies that

$$\begin{aligned}\|t_n-x^*\|^2&\leq\|u_n-x^*\|^2-\|u_n-t_n\|^2+2\lambda_n\langle Ay_n,y_n-t_n\rangle\\&=\|u_n-x^*\|^2-\|(u_n-y_n)+(y_n-t_n)\|^2+2\lambda_n\langle Ay_n,y_n-t-n\rangle\\&=\|u_n-x^*\|^2-\|u_n-y_n\|^2-2\langle u_n-y_n,y_n-t_n\rangle\\&\quad-\|y_n-t_n\|^2+2\lambda_n\langle Ay_n,y_n-t_n\rangle\\&=\|u_n-x^*\|^2-\|u_n-y_n\|^2-\|y_n-t_n\|^2\\&\quad+2\langle u_n-\lambda_n Ay_n-y_n,t_n-y_n\rangle.\end{aligned}\tag{3.3}$$

From (2.1), we have

$$\langle u_n-\lambda_n Au_n-y_n,t_n-y_n\rangle\leq 0,$$

so that

$$\begin{aligned}\langle u_n-\lambda_n Ay_n-y_n,t_n-y_n\rangle&=\langle u_n-\lambda_n Au_n-y_n,t_n-y_n\rangle\\&\quad+\langle\lambda_n Au_n-\lambda_n Ay_n,t_n-y_n\rangle\\&\leq\langle\lambda_n Au_n-\lambda_n Ay_n,t_n-y_n\rangle\\&\leq\lambda_n\|Au_n-Ay_n\|\|t_n-y_n\|\\&\leq\lambda_n\frac{1}{\alpha}\|u_n-y_n\|\|t_n-y_n\|.\end{aligned}\tag{3.4}$$

Hence it follows from (3.3) and (3.4) that

$$\begin{aligned}\|t_n-x^*\|^2 &\le \|u_n-x^*\|^2-\|u_n-y_n\|^2-\|y_n-t_n\|^2\\ &\quad +2\lambda_n\frac{1}{\alpha}\|u_n-y_n\|\|t_n-y_n\|\\ &\le \|u_n-x^*\|^2-\|u_n-y_n\|^2-\|y_n-t_n\|^2\\ &\quad +\lambda_n\frac{1}{\alpha}(\|u_n-y_n\|^2+\|t_n-y_n\|^2)\\ &= \|u_n-x^*\|^2+(\lambda_n\frac{1}{\alpha}-1)\|u_n-y_n\|^2\\ &\quad +(\lambda_n\frac{1}{\alpha}-1)\|t_n-y_n\|^2.\end{aligned} \tag{3.5}$$

Since $\lambda_n \to 0$ as $n\to\infty$, there exists a positive integer N_0 such that $\lambda_n\frac{1}{\alpha}-1\le\frac{-1}{3}$, when $n\ge N_0$. Hence it follows from (3.5) that

$$\|t_n-x^*\|\le\|u_n-x^*\|. \tag{3.6}$$

Observe that

$$\begin{aligned}\|u_n-x^*\|^2 &= \|T_{r_n}(x_n-r_nAx_n)-T_{r_n}(x^*-r_nAx^*)\|^2\\ &\le \|x_n-r_nAx_n-(x^*-r_nAx^*)\|^2\\ &= \|x_n-x^*\|^2-2r_n\left\langle x_n-x^*, Ax_n-Ax^*\right\rangle+r_n^2\|Ax_n-Ax^*\|^2\\ &\le \|x_n-x^*\|^2-2\alpha r_n\|Ax_n-Ax^*\|^2+r_n^2\|Ax_n-Ax^*\|^2\\ &= \|x_n-x^*\|^2+r_n(r_n-2\alpha)\|Ax_n-Ax^*\|^2\\ &\le \|x_n-x^*\|^2.\end{aligned}$$

and hence

$$\|t_n-x^*\|\le\|x_n-x^*\|.$$

Thus, we can calculate

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_nf(x_n)+\beta_nx_n+\gamma_nS_nt_n-x^*\|\\ &\le \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|t_n-x^*\|\\ &\le \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\|\\ &\quad +\beta_n\|x_n-x^*\|+\gamma_n\|x_n-x^*\|\\ &\le (1-\alpha_n(1-\alpha))\|x_n-x^*\|^2+\alpha_n\|f(x^*)-x^*\|\\ &= (1-\alpha_n(1-\alpha))\|x_n-x^*\|^2\\ &\quad +\alpha_n(1-\alpha)\frac{\|f(x^*)-x^*\|}{1-\alpha}.\end{aligned}$$

It follows from induction that

$$\| x_n - x^* \| \le \max\{\| x_1 - x^* \|, \frac{\| f(x^*) - x^* \|}{1-\alpha}\}, \quad n \ge N_0.$$

Therefore, $\{x_n\}$ is bounded. Hence, so are $\{t_n\},\{S_n t_n\},\{Au_n\},\{Ay_n\}$, and $\{f(x_n)\}$.

Step 2 ($\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0$). We have

$$\begin{aligned}
\| t_{n+1} - t_n \| &\le \| P_C(u_{n+1} - \lambda_{n+1} Ay_{n+1}) - P_C(u_n - \lambda_n Ay_n) \| \\
&\le \| u_{n+1} - \lambda_{n+1} Ay_{n+1} - (u_n - \lambda_n Ay_n) \| \\
&= \| (u_{n+1} - \lambda_{n+1} Au_{n+1}) - (u_n - \lambda_{n+1} Au_n) \\
&\quad + \lambda_{n+1}(Au_{n+1} - Ay_{n+1} Au_n) + \lambda_n Ay_n \| \\
&\le \| (u_{n+1} - \lambda_{n+1} Au_{n+1}) - (u_n - \lambda_{n+1} Au_n) \| \\
&\quad + \lambda_{n+1}(\| Au_{n+1} \| + \| Ay_{n+1} \| + \| Au_n \|) + \lambda_n \| Ay_n \| \\
&\le (1 + \lambda_n \frac{1}{\alpha}) \| u_{n+1} - u_n \| \\
&\quad + \lambda_{n+1}(\| Au_{n+1} \| + \| Ay_{n+1} \| + \| Au_n \|) + \lambda_n \| Ay_n \|.
\end{aligned} \tag{3.7}$$

On the other hand, from $u_n = T_{r_n}(x_n - r_n Ax_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1} Ax_{n+1})$, we have

$$F(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \forall y \in C, \tag{3.8}$$

$$F(u_{n+1}, y) + \langle Ax_{n+1}, y - u_{n+1} \rangle + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \ge 0, \forall y \in C. \tag{3.9}$$

Putting $y = u_{n+1}$ in (3.8) and $y = u_n$ in (3.9) and from (A2), we have

$$\langle Ax_n - Ax_{n+1}, u_{n+1} - u_n \rangle + \left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \ge 0$$

and hence

$$\langle Ax_n - Ax_{n+1}, u_{n+1} - u_n \rangle - \| u_{n+1} - u_n \|^2 + \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) \rangle \ge 0.$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c$ for all $n \in N$. Then, we have

$$\begin{aligned}
\| u_{n+1} - u_n \|^2 &\le \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) \rangle \\
&\quad + \frac{1}{\alpha} \| x_n - x_{n+1} \| \| u_{n+1} - u_n \| \\
&\le \| u_{n+1} - u_n \| (1 + \frac{1}{\alpha}) \| x_{n+1} - x_n \| \\
&\quad + | 1 - \frac{r_n}{r_{n+1}} | \| u_{n+1} - x_{n+1} \|
\end{aligned}$$

and hence

$$\begin{aligned}\| u_{n+1}-u_n \| &\leq (1+\frac{1}{\alpha})\| x_{n+1}-x_n \| + \frac{1}{r_{n+1}}| r_{n+1}-r_n | \| u_{n+1}-x_{n+1} \| \\ &\leq (1+\frac{1}{\alpha})\| x_{n+1}-x_n \| + \frac{1}{c}| r_{n+1}-r_n | M\end{aligned} \tag{3.10}$$

where $M = \sup\{\| u_n - x_n \| : n \in N\}$. It follows from (3.7) and the last inequality that

$$\begin{aligned}\| t_{n+1}-t_n \| \leq{}& (1+\frac{2\lambda_{n+1}}{\alpha^2})\| x_{n+1}-x_n \| + (1+\frac{\lambda_{n+1}}{\alpha})\frac{1}{c}| r_{n+1}-r_n | M \\ &+\lambda_{n+1}(\| Au_{n+1} \| + \| Ay_{n+1} \| + \| Au_n \|) + \lambda_n \| Ay_n \|.\end{aligned} \tag{3.11}$$

Let $z_n = \dfrac{\alpha_n f(x_n) + \gamma_n S_n t_n}{1-\beta_n}$, we obtain $x_{n+1} = (1-\beta_n) z_n + \beta_n x_n$ for all $n \in N$. Thus, we have

$$\begin{aligned}\| z_{n+1}-z_n \| &= \| \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S_{n+1} t_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S_n t_n}{1-\beta_n} \| \\ &= \| \frac{\alpha_{n+1}}{1-\beta_{n+1}} f(x_{n+1}) + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (S_{n+1} t_{n+1} - S_n t_n) \\ &\quad - \frac{\alpha_n}{1-\beta_n} f(x_n) - (1-\frac{\alpha_n}{1-\beta_n}) S_n t_n + (1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}) S_n t_n \| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \| f(x_{n+1}) - S_n t_n \| + \frac{\alpha_n}{1-\beta_n} \| S_n t_n - f(x_n) \| \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \| S_{n+1} t_{n+1} - S_n t_n \|.\end{aligned} \tag{3.12}$$

It follows from (3.11) that

$$\begin{aligned}\| S_{n+1} t_{n+1} - S_n t_n \| &\leq \| S_{n+1} t_{n+1} - S_{n+1} t_n \| + \| S_{n+1} t_n - S_n t_n \| \\ &\leq \| t_{n+1} - t_n \| + \| S_{n+1} t_n - S_n t_n \| \\ &\leq (1+\frac{2\lambda_{n+1}}{\alpha^2})\| x_{n+1}-x_n \| \\ &\quad + (1+\frac{\lambda_{n+1}}{\alpha})\frac{1}{c}| r_{n+1}-r_n | M + \lambda_n \| Ay_n \| \\ &\quad +\lambda_{n+1}(\| Au_{n+1} \| + \| Ay_{n+1} \| + \| Au_n \|) \\ &\quad + \| S_{n+1} t_n - S_n t_n \|.\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$\begin{aligned}\| z_{n+1}-z_n \| - \| x_{n+1}-x_n \| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}} \| f(x_{n+1}) - S_n t_n \| + \frac{\alpha_n}{1-\beta_n} \| S_n t_n - f(x_n) \| \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}} (1+\frac{2\lambda_{n+1}}{\alpha^2}) \| x_{n+1}-x_n \| \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}} (1+\frac{\lambda_{n+1}}{\alpha})\frac{1}{c}| r_{n+1}-r_n | M\end{aligned}$$

$$
\begin{aligned}
&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\lambda_{n+1}(\| Au_{n+1}\|+\| Ay_{n+1}\|+\| Au_n\|)\\
&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\lambda_n\| Ay_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\| S_{n+1}t_n-S_nt_n\|-\| x_{n+1}-x_n\|\\
\le&\frac{\alpha_{n+1}}{1-\beta_{n+1}}\| f(x_{n+1})-S_nt_n\|+\frac{\alpha_n}{1-\beta_n}\| S_nt_n-f(x_n)\|\\
&+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\frac{2\lambda_{n+1}}{\alpha^2}\| x_{n+1}-x_n\|\\
&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(1+\frac{\lambda_{n+1}}{\alpha})\frac{1}{c}| r_{n+1}-r_n| M\\
&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\lambda_{n+1}(\| Au_{n+1}\|+\| Ay_{n+1}\|+\| Au_n\|)\\
&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\lambda_n\| Ay_n\|\\
&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\sup\{\| S_{n+1}t-S_nt\|: t\in\{t_n\}\}.
\end{aligned}
$$

This together with (C1)-(C5) and $\lim_{n\to\infty}\sup\{\| S_{n+1}t-S_nt\|: t\in\{t_n\}\}=0$ implies that

$$\limsup_{n\to\infty}(\| z_{n+1}-z_n\|-\| x_{n+1}-x_n\|)\le 0.$$

Hence by Lemma2.2, we obtain $\| z_n-x_n\|\to 0$ as $n\to\infty$. It then follows that

$$\lim_{n\to\infty}\| x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\| z_n-x_n\|=0. \tag{3.14}$$

By (3.10) and (3.11), we also have

$$\lim_{n\to\infty}\| t_{n+1}-t_n\|=\lim_{n\to\infty}\| u_{n+1}-u_n\|=0.$$

Step 3 ($\lim_{n\to\infty}\| St_n-t_n\|=0$). Indeed, we take any $x^*\in\bigcap F(S)\bigcap VI(A,C)\bigcap EP$, to obtain

$$
\begin{aligned}
\| u_n-x^*\|^2=&\| T_{r_n}(x_n-r_nAx_n)-T_{r_n}(x^*-r_nAx^*)\|^2\\
\le&\left\langle u_n-x^*, x_n-x^*-r_n(Ax_n-Ax^*)\right\rangle\\
=&\left\langle u_n-x^*, x_n-x^*\right\rangle-r_n\left\langle u_n-x^*, Ax_n-Ax^*\right\rangle\\
\le&\left\langle u_n-x^*, x_n-x^*\right\rangle\\
=&\frac{1}{2}(\| u_n-x^*\|^2+\| x_n-x^*\|^2-\| x_n-u_n\|^2).
\end{aligned}
$$

Therefore, $\| u_n-x^*\|^2\le\| x_n-x^*\|^2-\| x_n-u_n\|^2$. From Lemma 2.1 and (3.6), we obtain, when $n\ge N_0$, that

$$\begin{aligned}\| x_{n+1} - x^* \|^2 &= \| \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n t_n - x^* \|^2 \\ &\le \alpha_n \| f(x_n) - x^* \|^2 + \beta_n \| x_n - x^* \|^2 + \gamma_n \| S_n t_n - x^* \|^2 \\ &\le \alpha_n \| f(x_n) - x^* \|^2 + \beta_n \| x_n - x^* \|^2 + \gamma_n \| t_n - x^* \|^2 \\ &\le \alpha_n \| f(x_n) - x^* \|^2 + \beta_n \| x_n - x^* \|^2 + \gamma_n \| u_n - x^* \|^2 \\ &\le \alpha_n \| f(x_n) - x^* \|^2 + \beta_n \| x_n - x^* \|^2 \\ &+ \gamma_n (\| x_n - x^* \|^2 - \| x_n - u_n \|^2) \\ &\le \alpha_n \| f(x_n) - x^* \|^2 + (1-\alpha_n) \| x_n - x^* \|^2 - \gamma_n \| x_n - u_n \|^2\end{aligned}$$

and hence

$$\begin{aligned}\gamma_n \| x_n - u_n \|^2 &\le \alpha_n \| f(x_n) - x^* \|^2 + \| x_n - x^* \|^2 - \| x_{n+1} - x^* \|^2 \\ &\le \alpha_n \| f(x_n) - x^* \|^2 \\ &\quad + \| x_n - x_{n+1} \| (\| x_n - x^* \| + \| x_{n+1} - x^* \|).\end{aligned}$$

It now follows from the last inequality, (C1), (C2), (C3) and (3.14), that

$$\lim_{n\to\infty} \| x_n - u_n \| = 0.$$

Noting that

$$\| y_n - x_n \| = \| P_C(u_n - \lambda_n A u_n) - x_n \| \le \| u_n - x_n \| + \lambda_n \| A u_n \| \to 0$$
$$\| y_n - t_n \| = \| P_C(u_n - \lambda_n A u_n) - P_C(u_n - \lambda_n A y_n) \| \le \lambda_n \| A u_n - A y_n \| \to 0$$

Thus

$$\| t_n - x_n \| \le \| t_n - y_n \| + \| y_n - x_n \| \to 0 \ as \ n \to \infty. \tag{3.15}$$

We note that

$$\begin{aligned}\| S_n y_n - x_{n+1} \| &\le \| S_n y_n - S_n t_n \| + \| S_n t_n - x_{n+1} \| \\ &\le \| y_n - t_n \| + \alpha_n \| S_n - f(x_n) \| + \beta_n \| S_n t_n - x_n \| \\ &\le \| y_n - t_n \| + \alpha_n \| S_n - f(x_n) \| \\ &\quad + \beta_n (\| S_n t_n - S_n x_n \| + \| S_n x_n - x_n \|) \\ &\le \| y_n - t_n \| + \alpha_n \| S_n - f(x_n) \| \\ &\quad + \beta_n (\| t_n - x_n \| + \| S_n x_n - x_n \|).\end{aligned} \tag{3.16}$$

Using (3.16), we have

$$\begin{aligned}\| S_n x_n - x_n \| &\le \| S_n x_n - S_n y_n \| + \| S_n y_n - x_{n+1} \| + \| x_{n+1} - x_n \| \\ &\le \| x_n - y_n \| + \| y_n - t_n \| + \alpha_n \| S_n t_n - f(x_n) \| \\ &\quad + \beta_n \| t_n - x_n \| + \beta_n \| S_n x_n - x_n \| + \| x_{n+1} - x_n \|,\end{aligned}$$

so that

$$(1-\beta_n) \| S_n x_n - x_n \| \le \| x_n - y_n \| + \| y_n - t_n \| + \alpha_n \| S_n t_n - f(x_n) \| + \beta_n \| t_n - x_n \| + \| x_{n+1} - x_n \|.$$

This implies that

$$\lim_{n\to\infty} \| S_n x_n - x_n \| = 0. \tag{3.17}$$

It follows from (3.15) and (3.17) that

$$\begin{aligned} \| S_n t_n - t_n \| &\le \| S_n t_n - S_n x_n \| + \| S_n x_n - x_n \| + \| x_n - t_n \| \\ &\le 2 \| t_n - x_n \| + \| S_n x_n - x_n \| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.18}$$

Applying Lemma 2.4 and (3.18), we have

$$\begin{aligned} \| St_n - t_n \| &\le \| St_n - S_n t_n \| + \| S_n t_n - t_n \| \\ &\le \sup\{\| St - S_n t \| : t \in \{t_n\}\} + \| S_n t_n - t_n \| \to 0. \end{aligned} \tag{3.19}$$

It follows from the last inequality and (3.15) that

$$\| x_n - St_n \| \le \| x_n - t_n \| + \| t_n - St_n \| \to 0 \text{ as } n \to \infty.$$

Step 4 ($\limsup_{n\to\infty} \langle f(q) - q, x_n - q \rangle \le 0$). Since $\{t_n\}$ is bounded, there exists a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \rightharpoonup z$. Since C is closed and convex, C is weakly closed. So, we have $z \in C$. Let us show that $z \in \bigcap F(S) \bigcap VI(A,C) \bigcap EP$. we first show $z \in EP$. Since $u_n = T_{r_n}(x_n - r_n A x_n)$, for any $y \in C$ we have

$$F(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0.$$

From(A2), we have

$$\langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge F(y, u_n).$$

Replacing n by n_i, we have

$$\langle Ax_{n_i}, y - u_{n_i} \rangle + \frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \ge F(y, u_{n_i}). \tag{3.20}$$

Put $y_t = ty + (1-t)z$ for all $t \in (0,1]$ and $y \in C$. Then, we have $y_t \in C$. So, from (3.20) we have

$$\begin{aligned} \langle y_t - u_{n_i}, Ay_t \rangle &\ge \langle y_t - u_{n_i}, Ay_t \rangle y_t - u_{n_i}, Ax_{n_i} \rangle \\ &\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle + F(y_t, u_{n_j}). \\ &= \langle y_t - u_{n_i}, Ay_t - Au_{n_i} \rangle + \langle y_t - u_{n_i}, Au_{n_i} - Ax_{n_i} \rangle \\ &\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle + F(y_t, u_{n_j}). \end{aligned}$$

Since $\| x_{n_i} - u_{n_i} \| \to 0$, we have $\| Ax_{n_i} - Au_{n_i} \| \to 0$. Further, from monotonicity of A, we have $\langle y_t - u_{n_i}, Ay_t - AU_{n_i} \rangle \geq 0$. So, from (A4) we have

$$\langle y_t - z, Ay_t \rangle \geq F(y_t, z), i \to \infty. \tag{3.21}$$

From (A1), (A4) and (3.20), we also have

$$\begin{aligned} 0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, z) \\ &\leq tF(y_t, y) + (1-t)\langle y_t - z, Ay_t \rangle \\ &= tF(y_t, y) + (1-t)t\langle y - z, Ay_t \rangle \end{aligned}$$

and hence

$$F(y_t, y) + (1-t)\langle y - z, Ay_t \rangle \geq 0.$$

Let $t \to 0$, we have, for each $y \in C$,

$$F(z, y) + \langle y - z, Az \rangle \geq 0.$$

This implies $z \in EP$. By the Opial's condition, we can obtain that $z \in F(S)$. Next we will show that $z \in VI(A, C)$. let

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v, \omega) \in G(T)$. Since $\omega - Av \in N_C v$ and $t_n \in C$, we have $\langle v - t_n, \omega - Av \rangle \geq 0$. On the other hand, from $t_n = P_C(u_n - \lambda_n Ay_n)$, we have

$$\langle v - t_n, t_n - (u_n - \lambda_n Ay_n) \rangle \geq 0,$$

that is,

$$\langle v - t_n, \frac{t_n - u_n}{\lambda_n} + Ay_n \rangle \geq 0.$$

Therefore, we obtain

$$\begin{aligned} \langle v - t_{n_i}, \omega \rangle &\geq \langle v - t_{n_i}, Av \rangle \\ &\geq \langle v - t_{n_i}, Av \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \rangle \\ &= \langle v - t_{n_i}, Av - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &= \langle v - t_{n_i}, Av - Ay_{n_i} \rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle. \end{aligned} \tag{3.22}$$

Noting that $\| t_{n_i} - y_{n_i} \| \to 0$, $\| t_{n_i} - u_{n_i} \| \to 0$ as $n \to \infty$, A is Lipschitz continuous and (3.21), we obtain

$$\langle v - z, \omega \rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$, and hence $z \in VI(A, C)$. Hence $z \in F(S) \cap VI(A, C) \cap EP$. It follows from (3.15) that $x_{n_i} \rightharpoonup z$, so

$$\begin{aligned}\limsup_{n\to\infty} \langle f(q) - q, x_n - q \rangle &= \lim_{n\to\infty} \langle f(q) - q, x_{n_i} - q \rangle \\ &= \langle f(q) - q, z - q \rangle \leq 0.\end{aligned}$$

Step 5 $(\lim_{n\to\infty} \| x_n - q \| = 0)$. Indeed, we observe that

$$\begin{aligned}\| x_{n+1} - q \|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n t_n - q, x_{n+1} - q \rangle \\ &= \alpha_n \langle f(x_n) - q, x_{n+1} - q \rangle + \beta_n \langle x_n - q, x_{n+1} - q \rangle \\ &\quad + \gamma_n \langle S_n t_n - q, x_{n+1} - q \rangle \\ &\leq \frac{1}{2} \beta_n (\| x_n - q \|^2 + \| x_{n+1} - q \|^2) \\ &\quad + \frac{1}{2} \gamma_n (\| t_n - q \|^2 + \| x_{n+1} - q \|^2) \\ &\quad + \alpha_n \langle f(x_n) - f(q), x_{n+1} - q \rangle + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\ &\leq \frac{1}{2} (1 - \alpha_n)(\| x_n - q \|^2 + \| x_{n+1} - q \|^2) \\ &\quad + \frac{1}{2} \alpha_n (\| f(x_n) - f(q) \|^2 + \| x_{n+1} - q \|^2) + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\ &\leq \frac{1}{2} (1 - \alpha_n (1 - \alpha^2)) \| x_n - q \|^2 + \frac{1}{2} (1 - \alpha_n) \| x_{n+1} - q \|^2 \\ &\quad + \frac{1}{2} \alpha_n \| x_{n+1} - q \|^2 + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle\end{aligned}$$

which implies that

$$\| x_{n+1} - q \|^2 \leq (1 - \alpha_n (1 - \alpha^2)) \| x_n - q \|^2 + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle. \tag{3.23}$$

Setting $\delta_n = 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle$, we have $\limsup_{n\to\infty} \dfrac{\delta_n}{\alpha_n (1 - \alpha^2)} \leq 0$. Applying Lemma 2.3 to (3.23), we conclude that $\{x_n\}$ converges strongly to q. Consequently, $\{u_n\}, \{y_n\}$ converges strongly to q. This completes the proof.

As in [9, Theorem 4.1], we can generate a sequence $\{S_n\}$ of nonexpansive mappings satisfying condition $\sum_{n=1}^{\infty} \sup\{\| S_{n+1} z - S_n z \| : z \in B\} < \infty$ for any bounded subset B of C by using convex combination of a general sequence $\{T_k\}$ of nonexpansive mappings with a common fixed point.

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ into R satisfying (A1)-(A4), $A: C\to H$ an α-inverse-strongly monotone mapping and let $\beta_n^{\ k}$ be a family of nonnegative numbers with indices $n,k\in N$ with $k\le n$ such that*

(i) $\sum_{k=1}^{n}\beta_n^{\ k}=1$ *for all* $n\in N$;
(ii) $\lim_{n\to\infty}\beta_n^{\ k}>0$ *for every* $k\in N$;
(iii) $\sum_{n=1}^{\infty}\sum_{k=1}^{n}|\beta_{n+1}^{k}-\beta_n^{k}|<\infty$. *Let* $\{T_k\}$ *ba a sequence of nonexpansive mappings of C into itself such that* $\bigcap_{n=1}^{\infty}F(T_k)\bigcap VI(A,C)\bigcap EP\neq\emptyset$. *Let the sequence* $\{x_n\},\{u_n\},\{y_n\}$ *be generated by*

$$x_1=x\in C \quad \text{chosen arbitrary},$$
$$F(u_n,y)+\langle Ax_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\forall y\in C,$$
$$y_n=P_C(u_n-\lambda_n Au_n),$$
$$x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n\sum_{k=1}^{n}\beta_n^{\ k}T_kP_C(u_n-\lambda_n Ay_n),\ n\ge 1,$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subseteq[0,1]$, $\{\lambda_n\}\subset(0,1)$, *and* $\{r_n\}\subset(0,2\alpha]$ *satisfy the following conditions:*

(C1) $\alpha_n+\beta_n+\gamma_n=1$,
(C2) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}=\infty$,
(C3) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$,
(C4) $\lim_{n\to\infty}\lambda_n=0$,
(C5) $\liminf_{n\to\infty}r_n>0$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$.

Then the sequences $\{x_n\},\{y_n\},\{u_n\}$ *converge strongly to the same point q, where* $q=P_{\bigcap_{n=1}^{\infty}F(T_k)\bigcap VI(A,C)\bigcap EP}f(q)$.

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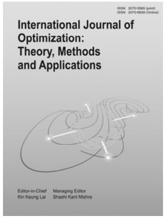

New Type of Second Order Duals for A Nonlinear Programming Problem

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Abstract. First order duality results for a nonlinear programming problem already exist in optimization theory. Study of second order dual is important due to its computational advantage. In this paper second order Lagrange dual, Fenchel dual and Fenchel Lagrange dual are constructed. Weak and strong duality results are proved and duality relationship between these three second order duals under appropriate assumptions are established.

Keywords: second order dual, conjugate function, Lagrange first order dual, Fenchel first order dual.

1 Introduction

The concept of duality plays an important role to study the existence of the solution of nonlinear programming problems. Different types of dual problems are associated with a primal nonlinear constrained optimization problem. The study of second order duality is significant due to the computational advantage over first order dual as its provides tighter bounds for the value of the objective function when approximations are used ([1], [2]). One of the most fruitful theories of duality in convex optimization bases on the concept of conjugate functions. This concept was due to Fenchel [3] and Rockfellar [4] in the finite dimensional case. Wanka and Bot [5], [6] used this flexible theory to an optimization problem with inequality and cone inequality constraints in a finite di-

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mensional space and discussed the duality results for general first order Lagrange dual, Fenchel dual and Fenchel-Lagrange dual problems under convexity assumptions.

Mangasarian [1] extended the duality concept of a nonlinear primal problem by taking first and second order approximation of the objective function and constraint functions. Hanson and Mond [7], [8] studied the weak and strong duality for Wolfe first order dual and Mond Weir first order dual with generalized convexity assumptions. In both cases the duals are constrained optimization problems. In this paper second order Lagrange, Fenchel and Fenchel-Lagrange duals are constructed which are unconstrained optimization problems and duality results for these three second order duals are studied. Here we discuss some priliminary concepts which are used in sequel.

Let the primal nonlinear optimization problem be,

$$(P) \quad \inf_{x\in R^n} \quad f(x)$$
$$\text{subject to} \quad g(x) \leq 0,$$

where $f: R^n \to R$ and $g=(g_1, g_2, \cdots, g_m)^T : R^n \to R^m$ are twice differentiable functions. R is the extended real number system. Using the quadratic approximation of f and g about any feasible point $x \in R^n$ of (P) can be presented as

$$(P^2) \quad \inf_{x,\, x\in R^n} \quad f(x)+r^T\nabla_x f(x)+\frac{1}{2}r^T\nabla_x^2 f(x)r$$
$$\text{subject to } g(x)+r^T\nabla_x g(x)+\frac{1}{2}r^T\nabla_x^2 g(x)r \leq 0$$

We denote $F_2(x,r)=f(x)+r^T\nabla_x f(x)+\frac{1}{2}r^T\nabla_x^2 f(x)r$ and $G_2(x,r)=g(x)+r^T\nabla_x g(x)+\frac{1}{2}r^T\nabla_x^2 g(x)r.$

Definition 1.1 *[4] The function $f^*: R^n \to R,$ defined by*

$$f^*(p^*)\sup_{x\in R^n}\{p^{*T}x-f(x)\}$$

is called as the conjugate function of f: $R^n \to R$.

Lemma 1.1 *[4] Let C_1 and C_2 be non-empty sets in R^n. There exists a hyperplane separating C_1 and C_2 properly if and only if there exists a vector b such that*

(a) $\inf\ \{x^T b \mid x\in C_1\} \geq \sup\{x^T b \mid x \in C_2\},$

(b) $\sup\ \{x^T b \mid x\in C_1\} > \text{jnf}\{x^T b \mid x \in C_2\}.$

There exists a hyperplane separating C_1 and C_2 strongly if and only if there exists a vector b such that

(c) $\inf\{x^T b \mid x\in C_1\} > \sup\{x^T b \mid x \in C_2\}.$

Lemma 1.2 *[4] Let C_1 and C_2 be non-empty convex sets in R^n. In order that there exists a hyperplane separating C_1 and C_2 properly, it is necessary and sufficient that $\text{rint}\, C_1$ and $\text{rint}\, C_2$ have no point in common. ($\text{rint}\, C_1$ and $\text{rint}\, C_2$ are the relative interiors of C_1 and C_2)*

Lemma 1.3 *[9] Let X be a non-empty convex set in* R^n. *Let* $\alpha : R^n \to R$ *and* $g : R^n \to R^m$ *be convex, and let* $h : R^n \to R$ *be affine; that is,* $h = Ax - b$. *If System 1 below has no solution x, then System 2 has a solution* (u_0, u, v). *The converse holds if* $u_0 > 0$.

System 1: $\alpha(x) < 0 \quad g(x) \le 0 \quad h(x) = 0$ *for some* $x \in X$
System 2: $u_0 \alpha(x) + u^T g(x) + v^T h(x) \ge 0$ *for all* $x \in X$
$(u_0, u) \ge 0, \quad (u_0, u, v) \ne 0$

2 Construction of Different Types of Second Order Duals Using Conjugate Functions

This section deals with the construction of unconstrained second order Lagrange dual, Fenchel dual and Fenchel Lagrange dual of a general nonlinear programming problem (P^2) using different types of conjugate functions.

2.1 Second Order Lagrange Dual

Let the function $\Phi_L^2 : R^n \times R^n \times R^m \to R$ be

$$\Phi_L^2(x, r, q) = \begin{cases} F_2(x, r) & \text{if } G_2(x, r) \le q,\ x, r \in R^n,\ q \in R^m \\ \infty & \text{otherwise} \end{cases}$$

where $F_2(x, r)$ and $G_2(x, r)$ are defined in section 1. Using Definition 1.1, the conjugate of Φ_L^2 is Φ_L^{*2} which is defined as
$\Phi_L^{*2} : R^n \times R^n \times R^m \to R$ such that,

$$\Phi_L^{*2}(x^*, r^*, q^*) = \sup_{x, r \in R^n,\ q \in R^m} [x^{*T} x + r^{*T} r + q^{*T} q - \Phi_L^2(x, r, q)]$$

$$= \sup_{x, r \in R^n,\ q \in R^m,\ G_2(x, r) \le q} [x^{*T} x + r^{*T} r + q^{*T} q - F_2(x, r)]$$

If $s = q - G_2(x, r)$ then,

$$\Phi_L^{*2}(x^*, r^*, q^*) = \sup_{x, r \in R^n,\ S \in R^m,\ s \ge 0} [x^{*T} x + r^{*T} r + q^{*T} s + q^{*T} G_2(x, r) - F_2(x, r)]$$

$$= \sup_{x, r \in R^n} [x^{*T} x + r^{*T} r + q^{*T} G_2(x, r) - F_2(x, r)] + \sup_{s \in R^m,\ s \ge 0} q^{*T} s$$

If $q^* \le 0$ then $\sup_{s \in R^m,\ s \ge 0} \quad q^{*T} s = 0$

$$\Phi_L^{*2}(x^*, r^*, q^*) = \sup_{x, r \in R^n} [x^{*T} x + r^{*T} r + q^{*T} G_2(x, r) - F_2(x, r)]$$

$$\text{and } \Phi_L^{*2}(0, 0, q^*) = \sup_{x, r \in R^n} [q^{*T} G_2(x, r) - F_2(x, r)]$$

$$\text{Here} -\Phi_L^{*2}(0, 0, q^*) = \inf_{x,\, r\in R^n} [F_2(x, r) + q^{*T} G_2(x, r)]$$

The Lagrange dual of (P^2) takes the following form. We denote it by D_L^2 which means Lagrange second order dual of (P^2) at any point $x \in R^n$

$$(D_L^2) \sup_{q^*\geq 0}[-\Phi_L^{*2}(0, 0, q^*)] = \sup_{q^*\geq 0} \inf_{x,\, r\in R^n} [F_2(x, r) + q^{*T} G_2(x, r)]$$
$$= \sup_{q^*\geq 0} \inf_{x,\, r\in R^n} \; L(x, r, q^*)$$

where $L(x, r, q^*)$ is the Lagrange function.

2.2 Second Order Fenchel Dual

Let the function $\Phi_F^2 : R^n \times R^n \times R^n \times R^n \to R$ be

$$\Phi_F^2(x, r, p_1, p_2) = \begin{cases} F_2(x + p_1, r + p_2) & \text{if } G_2(x, r) \leq 0,\, x, r \in R^n,\, p_1, p_2 \in R^n \\ \infty & \text{otherwise} \end{cases}$$

where $F_2(x, r)$ and $G_2(x, r)$ are defined in section 1. Using Definition 1.1, the conjugate of Φ_F^2 is $\Phi_F^{*2} : R^n \times R^n \times R^n \times R^n \to R$ efined by

$$\Phi_F^{*2}(x^*, r^*, p_1^*, p_2^*) = \sup_{p_1,\, p_2,\, x,\, r\in R^n} [x^{*T} x + r^{*T} r + p_1^{*T} p_1 + p_2^{*T} p_2 - \Phi_F^2(x, r, p_1, p_2)]$$
$$= \sup_{p_1,\, p_2,\, x,\, r\in R^n,\, G_2(x, r)\leq 0} [x^{*T} x + r^{*T} r + p_1^{*T} p_1 + p_2^{*T} p_2 - F_2(x + p_1, r + p_2)]$$

If $x + p_1 = s, r + p_2 = t,$ then

$$\Phi_F^{*2}(0, 0, p_1^*, p_2^*) = F_2^*(p_1^*, p_2^*) + \sup_{x,\, r\in R^n,\, G_2(x, r)\leq 0} [-p_1^{*T} x - p_2^{*T} r]$$

The second order Fenchel type dual of (P^2) at $x \in R^n$ denoted by (D_F^2) can be expressed as,

$$(D_F^2) \sup_{p_1^*,\, p_2^*\in R^n} [-\Phi_F^*(0, 0, p_1^*, p_2^*)] = \sup_{p_1^*,\, p_2^*\in R^n} [-F_2^*(p_1^*, p_2^*) + \inf_{x,\, r\in R^n,\, G(x, r)\leq 0} [p_1^{*T} x + p_2^{*T} r]$$

2.3 Second Order Fenchel-Lagrange Dual

Let the function $\Phi_{FL}^2 : R^n \times R^n \times R^n \times R^n \times R^m \to R$ be

$$\Phi_{FL}^2(x, r, p_1, p_2, q) = \begin{cases} F_2(x + p_1, r + p_2) & \text{if } G_2(x, r) \leq q,\, x, r \in R^n,\, p_1, p_2 \in R^n,\, q \in R^m \\ \infty & \text{otherwise} \end{cases}$$

where $F_2(x, r)$ and $G_2(x, r)$ are defined in section 1. Using Definition 1.1, the conjugate of Φ_{FL}^2 is $\Phi_{FL}^{*2}: R^n \times R^n \times R^n \times R^n \times R^m \to R$

$$\Phi_{FL}^{*2}(x^*, r^*, p_1^*, p_2^*, q^*) = \sup_{p_1,\, p_2,\, x,\, r \in R^n,\, q \in R^m} [x^{*T}x + r^{*T}r + p_1^{*T}p_1 + p_2^{*T}p_2 + q^{*T}q - \Phi_{FL}^2(x, r, p_1, p_2, q)]$$
$$= \sup_{p_1,\, p_2,\, x,\, r \in R^n,\, q \in R^m,\, G_2(x, r) \le q} [x^{*T}x + r^{*T}r + p_1^{*T}p_1 + p_2^{*T}p_2 + q^{*T}q - F_2(x + p_1, r + p_2)]$$

If $x + p_1 = s, r + p_2 = t, w = q - G_2(x, r)$then

$$\Phi_{FL}^{*2}(x^*, r^*, p_1^*, p_2^*, q^*) = \sup_{x,\, t \in R^n} [p_1^{*T}s + p_2^{*T}t - F(s, t)]$$
$$+ \sup_{x,\, r \in R^n} [(x^{*T} - p_1^{*T})x + (x^{*T} - p_2^{*T})r + q^{*T}G_2(x, r)]$$
$$+ \sup_{w \in R^m,\, w \ge 0} q^{*T}w$$

If $q^* \le 0$ then $\sup_{w \in R^m,\, w \ge 0} q^{*T}w = 0.$ So for $q^* \le 0,$

$$\Phi_{FL}^{2*}(x^*, r^*, p_1^*, p_2^*, q^*) = \sup_{s,\, t \in R^n} [p_1^{*T}s + p_2^{*T}t - F_2(s, t)]$$
$$+ \sup_{x,\, r \in R^n} [(x^{*T} - p_1^{*T})x + (x^{*T} - p_2^{*T})r + q^{*T}G_2(x, r)]$$

Let (D_{FL}^2) denotes the second order Fenchel Lagrange type dual of (P^2) at $x \in R^n$ which is,

$$(D_{FL}^2) \quad \sup_{p_1^*,\, p_2^* \in R^n,\, q^* \ge 0} [-\Phi_{FL}^{*2}(0, 0, p_1^*, p_2^*, q^*) = \sup_{p_1^*,\, p_2^* \in R^n,\, q^* \ge 0} [-F_2^*(p_1^*, p_2^*)$$
$$+ \inf_{x,\, r \in R^n} [p_1^{*T}x + p_2^{*T}r + q^{*T}G_2(x, r)]]$$

Note: Throughout the paper we denote $\sup(D_L^2)$, $\sup(D_F^2)$, $\sup(D_{FL}^2)$, $\inf(P^2)$, $\inf(P)$ for the optimal obejctive values of the problems (D_L^2), (D_F^2), (D_{FL}^2), (P^2), (P) respectively.

3 Some Duality Results between (P) and (D_L^2), (D_F^2), (D_{FL}^2)

In this section we prove that the optimal values of the Lagrange dual problem (D_L^2), the Fenchel dual problem (D_F^2) and the Fenchel Lagrange dual problem (D_{FL}^2) are equal under certain conditions and weak duality and strong duality results hold for the original primal (P) and these second order duals.

Theorem 3.1 *The following inequalities hold between (P),* (P^2), (D_L^2), (D_F^2), (D_{FL}^2).

1. $\sup(D_L^2) \le \inf(P^2) \le \inf(P)$
2. $\sup(D_F^2) \le \inf(P^2) \le \inf(P)$
3. $\sup(D_{FL}^2) \le \inf(P^2) \le \inf(P)$.

Proof: (1) From section 2.1,

$$\Phi_L^{*2}(0, 0, q^*) = \sup_{x,\, r\in R^n,\, q\in R^m} [q^{*T} q - \Phi_L^2(x, r, q)]$$
$$\geq \sup_{x,\, r\in R^n} \left\{q^{*T} 0 - \Phi_L^2(x, r, 0)\right\} = \sup_{x,\, r\in R^n} \left\{-\Phi_L^2(x, r, 0)\right\}$$

i.e., for each $x, r \in R^n$ and $q^* \in R^m$, $-\Phi_L^{*2}(0, 0, q^*) \leq \Phi_L^2(x, r, 0)$, which implies that

$$\sup_{q^*\geq 0}\{-\Phi_L^{*2}(0, 0, q^*)\} \leq \inf_{x,\, r\in R^n} \{\Phi_L^2(x, r, 0)\} \text{ i.e. } \sup(D_L^2) \leq \inf(P^2).$$

Also $$\inf_{x,\, r\in R^n} \{\Phi_L^2(x, r, 0)\} \leq \inf_{x\in R^n} \{\Phi_L^2(x, 0, 0)\} \text{ i.e. } \inf(P^2) \leq \inf(P),$$

which implies that $\sup(D_L^2) \leq \inf(P^2) \leq \inf(P)$. Using the conjugate functions D_F^{*2} and D_{FL}^{*2} the inequalities (2) and (3) can be proved.

Thus the optimal objective values of the dual problems (D_L^2), (D_F^2) and (D_{FL}^2) are less than or equal to the optimal objective value of the primal problem (P^2) and (P).

Theorem 3.2 *The inequality* $\sup(D_L^2) \geq \sup(D_{FL}^2)$ *holds.*

Proof: Let $q^* \geq 0$ and $p_1^*, p_2^* \in R^n$ be fixed. By the definition of the conjugate function, for each $x, r \in R^n$,

$$F_2^*(p_1^*, p_2^*) \geq p_1^{*T} x + p_2^{*T} r - F_2(x, r)$$

For each $x, r \in R^n$,

$$F_2(x, r) + q^{*T} G_2(x, r) \geq -F_2^*(p_1^*, p_2^*) + p_1^{*T} x + p_2^{*T} r + q^{*T} G_2(x, r)$$

This means for all $q^* \geq 0$ and $p_1^*, p_2^* \in R^n$,

$$\inf_{x,\, r\in R^n} [F_2(x, r) + q^{*T} G_2(x, r)] \geq -F_2^*(p_1^*, p_2^*) + \inf_{x,\, r\in R^n} [p_1^{*T} x + p_2^{*T} r + q^{*T} G_2(x, r)] \tag{1}$$

Taking supremum over $q^* \geq 0$ and $p_1^*, p_2^* \in R^n$ this becomes

$$\sup_{q^*\geq 0} \inf_{x,\, r\in R^n} [F_2(x, r) + q^{*T} G_2(x, r)] \geq \sup_{p_1^*, p_2^*\in R^n, q^*\geq 0} \{-F_2^*(p_1^*, p_2^*) + \inf_{x,\, r\in R^n} [p_1^{*T} x + p_2^{*T} r + q^{*T} G_2(x, r)]\}.$$

The last inequality is in fact $\sup(D_L^2) \geq \sup(D_{FL}^2)$ and the proof is complete.

The following example shows that the inequality in the theorem above may be fulfilled strictly and the second order dual gives a tighter bound than the original dual.

Example 3.1 *Let be* $X = [0, \infty)$, $f : R^2 \to R$,

$$F(x, y) = \begin{cases} -x^2 - y^2 & \text{if } x, y \in X \\ +\infty & \text{otherwise} \end{cases}$$

$g: R \to R,\ g(x) = x^2 + y^2 - 4.$ *A straight forward calculation show that the supremum of the second order Lagrange dual is* $\sup(D_L^2) = -4$. *On the other hand for the second order Fenchel-Lagrange dual,* $\sup(D_{FL}^2) = -\infty$. *Hence we have seen that* $\sup(D_{FL}^2) \le \sup(D_L^2)$.

Theorem 3.3 *The inequality* $\sup(D_L^2) \le \sup(D_{FL}^2)$ *holds.*

Proof: Let $p_1^*, p_2^* \in R^n$ be fixed. For each $q^* \ge 0$ we have

$$\inf_{x,\, r \in R^n} [p_1^{*T} x + p_2^{*T} r + q^{*T} G_2(x, r)] \le \inf_{x,\, r \in R^n, G_2(x, r) \le 0} [p_1^{*T} x + p_2^{*T} r + q^{*T} G_2(x, r)]$$
$$\le \inf_{x,\, r \in R^n, G_2(x, r) \le 0} p_1^{*T} x + p_2^{*T} r.$$

Then for every $p_1^*, p_2^* \in R^n$,

$$\sup_{q^* \ge 0} \inf_{x,\, r \in R^n} [p_1^{*T} x + p_2^{*T} r + q^{*T} G_2(x, r)] \le \inf_{x,\, r \in R^n, G_2(x, r) \le 0} [p_1^{*T} x + p_2^{*T} r]. \tag{2}$$

By adding $-F_2^*(p_1^*, p_2^*)$ to both sides we obtain

$$-F_2^*(p_1^*, p_2^*) + \sup_{q^* \ge 0} \inf_{x,\, r \in R^n} [p_1^{*T} x + p_2^{*T} r + q^{*T} G_2(x, r)]$$
$$\le -F_2^*(p_1^*, p_2^*) + \inf_{x,\, r \in R^n, G_2(x, r) \le 0} p_1^{*T} x + p_2^{*T} r, \text{ for all } p_1^*, p_2^* \in R^n.$$

This inequality implies

$$\sup_{p_1^*,\, p_2^* \in R^n,\, q^* \ge 0} \{-F_2^*(p_1^*, p_2^*) + \inf_{x,\, r \in R^n} [p_1^{*T} x + p_2^{*T} r + q^{*T} G_2(x, r)]\}$$
$$\le \sup_{p_1^*,\, p_2^* \in R^n} \{-F_2^*(p_1^*, p_2^*) + \inf_{x,\, r \in R^n, G_2(x, r) \le 0} p_1^{*T} x + p_2^{*T} r\}$$

or equivalently, $\sup(D_F^2) \ge \sup(D_{FL}^2)$ and the proof is complete.

Theorem 3.4 *If f and* $g_i, i = 1, \cdots, m$ *are convex fuctions then* $\sup(D_L^2) = \sup(D_{FL}^2)$.

Proof: Let $q^* \in R^m, q^* \ge 0$ be fixed and define $\alpha = \inf_{x,\, r \in R^n} [F_2(x, r) + q^* G_2(x, r)]$. It is necessary to show,

$$\inf_{x,\, r \in R^n} [F_2(x, r) + q^* G_2(x, r)] = \sup_{p_1^*,\, p_2^* \in R^n} \{-F_2^*(p_1^*, p_2^*) + \inf_{x,\, r \in R^n} [p_1^{*T} x + p_2^{*T} r + q^* G_2(x, r)]\}. \tag{3}$$

Theorem 3.2 implies

$$\alpha \ge \sup_{p_1^*,\, p_2^* \in R^n} \{-F_2^*(p_1^*, p_2^*) + \inf_{x,\, r \in R^n} [p_1^{*T} x + p_2^{*T} r + q^* G_2(x, r)]\}. \tag{4}$$

For $\alpha = -\infty$ the statement (3) is obvious. For $\alpha > -\infty$, consider the sets $C = epiF_2(x, r) = \{(x, r, \mu), x, r \in R^n, \mu \in R, F_2(x, r) \le \mu\} \subseteq R^{2n+1}$, (Here $epiF_2(x, r)$ is the epigraph of $F_2(x, r)$)

$$D = \{(x, r, \mu), x, r \in R^n, \mu \in R, \mu + q^* G_2(x, r) \le \alpha\} \subseteq R^{2n+1}.$$

Obviously $C, D \neq \emptyset$ are convex sets. Since f is convex, so $r^T \nabla_x^2 f(x) r$ is a positive semi definite quadratic form in r at x. So for every $x \in R^n$, $F_2(x, r)$ is a convex function of r at any $x \in R^n$. Hence the epigraph of $F_2(x, r)$ is a convex set i.e. C is convex set. Similarly since g is a convex function so $G_2(x, r)$ is a convex function of r at any $x \in R^n$. Since $q^* \geq 0$ so $\mu + q^* G_2(x, r)$ is a convex function. Hence D is a convex set. $(rintC) \bigcap D = \emptyset$ ($rintC$ is the relative interior of C), because $(rintC) \bigcap D \neq \emptyset$ causes a contradiction to the definition of α. Therefore C and D are properly separable by lemma 1.2. Hence by separation theorem (lemma 1.1), there exists $(p_1^*, p_2^*, \mu^*) \in R^n \times R^n \times R$, $(p_1^*, p_2^*, \mu^*) \neq (0, 0, 0)$ and $\alpha^* \in R$ such that the hyperplane $H = \{(x, r, \mu) : p_1^{*T} x + p_2^{*T} r + \mu^* \mu = \alpha^*\}$ separates C and D properly. Hence

$$\inf\{p_1^{*T} x + p_2^{*T} r + \mu^* \mu : (x, r, \mu) \in D\} \geq \alpha^* \geq \sup\{p_1^{*T} x + p_2^{*T} r + \mu^* \mu : (x, r, \mu) \in C\}, \tag{5}$$

$$\sup\{p_1^{*T} x + p_2^{*T} r + \mu^* \mu : (x, r, \mu) \in D\} > \inf\{p_1^{*T} x + p_2^{*T} r + \mu^* \mu : (x, r, \mu) \in C\}. \tag{6}$$

$\mu^* = 0$ implies $p_1^{*T} x + p_2^{*T} r = \alpha^*$ for all $x, r \in R^n$ using (5) which contradicts to strict inequality of (6). Let $\mu^* > 0$. In that case letting $\mu \to +\infty$. (5) is not possible since the supremum on the right hand side of (5) is $+\infty$, therefore $\alpha^* = +\infty$ which contradicts that α^* is finite. Therefore $\mu^* < 0$ holds. Dividing (5) by $(-\mu^*)$, (5) becomes

$$\inf\{p_{01}^{*T} x + p_{02}^{*T} r - \mu : (x, r, \mu) \in D\} \geq \alpha_0^* \geq \sup\{p_{01}^{*T} x + p_{02}^{*T} r - \mu : (x, r, \mu) \in C\},$$

where $\alpha_0^* = \alpha^* / (-\mu^*)$, $p_{01}^* = p_1^* / (-\mu^*)$ and $p_{02}^* = p_2^* / (-\mu^*)$. Hence

$$p_{01}^{*T} x + p_{02}^{*T} r - \mu \leq \alpha_0^* \quad \text{for all } (x, r, \mu) \in C \tag{7}$$

$$\alpha_0^* \leq p_{01}^{*T} x + p_{02}^{*T} r - \mu \quad \text{for all } (x, r, \mu) \in D \tag{8}$$

From (7) we have

$$p_{01}^{*T} x + p_{02}^{*T} r - F_2(x, r) \leq \alpha_0^* \quad \text{for all } x, r \in R^n \tag{9}$$

as a consequence of $(x, r, F_2(x, r)) \in C$. Calculating the supremum over $x, r \in R^n$ of left hand side of (9) we get,

$$F_2^*(p_{01}^*, p_{02}^*) \leq \alpha_0^*. \tag{10}$$

Furthermore $(x, r, \alpha - q^{*T} G_2(x, r)) \in D$ for all $x, r \in R^n$. Then by (8), $\alpha_0^* \leq p_{01}^{*T} x + p_{02}^{*T} r - (\alpha - q^{*T} G_2(x, r))$ for all $(x, r, \mu) \in R^n$, that is

$$\alpha_0^* + \alpha \leq \inf_{x, r \in R^n} [p_{01}^{*T} x + p_{02}^{*T} r + q^{*T} G_2(x, r)]$$

From (10), $\alpha \leq -F_2^*(p_{01}^*, p_{02}^*) + \inf_{x, r \in R^n} [p_{01}^{*T} x + p_{02}^{*T} r + q^{*T} G_2(x, r)]$. Using (4) it can be concluded that,

$$\alpha = -F_2^*(p_{01}^*, p_{02}^*) + \inf_{x, r \in R^n} [p_{01}^{*T} x + p_{02}^{*T} r + q^{*T} G_2(x, r)].$$

Hence (3) holds for each $q^* \in R^m, q^* \geq 0$. $\sup(D_L^2) = \sup(D_{FL}^2)$ follows taking supremum of both sides of (3) over $q^* \in R^m, q^* \geq 0$.

To prove, $\sup(D_L^2) = \sup(D_{FL}^2)$ we need constraint qualification (CQ).

(CQ): There exist elements $x', r' \in R^n$ such that $G_2(x', r') < 0$.

Theorem 3.5 *If* $g_i, i = 1, \cdots, m$ *are convex fuctions such that* $G = \{(x, r) : x, r \in R^n, G_2(x, r) \leq 0\} \neq \emptyset$ *and (CQ) is fulfilled then* $\sup(D_F^2) = \sup(D_{FL}^2)$.

Proof: For $p_1^*, p_2^* \in R^n$ fixed we first show that

$$\sup_{q^* \geq 0} \inf_{x,\, r \in R^n} [p_1^{*T} x + p_2^{*T} r + q^{*T} G_2(x, r)] = \inf_{x,\, r \in R^n,\, G_2(x, r) \leq 0} [p_1^{*T} x + p_2^{*T} r] \tag{11}$$

Since $g = (g_1, g_2, \cdots, g_m)$ is a convex function so $G_2(x, r)$ is a convex function of r at any $x \in R^n$, so infimum exist. $\beta = \inf_{x,\, r \in R^n,\, G(x, r) \leq 0} [p_1^{*T} x + p_2^{*T} r]$. If $\beta = -\infty$ then from (2) it follows that

$$\sup_{q^* \geq 0} \inf_{x,\, r \in R^n} [p_1^{*T} x + p_2^{*T} r + q^{*T} G_2(x, r)] = -\infty = \inf_{x,\, r \in R^n,\, G_2(x, r) \leq 0} [p_1^{*T} x + p_2^{*T} r].$$

Let $\beta \in (-\infty, +\infty)$. The system of inequalities $p_1^{*T} x + p_2^{*T} r - \beta < 0, G_2(x, r) \leq 0$ has no solution in R^n.

Using lemma 1.3 to the above system, it follows that there exists $u^* > 0, q^* \geq 0$ such that for each $x, r \in R^n, u^*(p_1^{*T} x + p_2^{*T} r - \beta) + q^{*T} G_2(x, r) \geq 0$. $u^* \neq 0$, because if $u^* = 0$ then $q^{*T} G_2(x, r) \geq 0$ and by (CQ) we get $x', r' \in R^n$ such that $q^{*T} G_2(x', r') < 0$ which is a contradiction.

Dividing the above inequality by u^* this becomes $p_1^{*T} x + p_2^{*T} r - \beta + q_0^{*T} G_2(x, r) \geq 0$, for all $x, r \in R^n$ where $q_0^* = q^* / u^*$. Hence

$$\inf_{x,\, r \in R^n} [p_1^{*T} x + p_2^{*T} r + q_0^{*T} G_2(x, r)] \geq \beta.$$

This last inequality and (2) imply (11). Adding $-F_2^*(p_1^*, p_2^*)$ to (11) and taking supremum subject to $p_1^*, p_2^* \in R^n$ we obtain $\sup(D_F^2) = \sup(D_{FL}^2)$.

Theorem 3.6 *(Strong duality) Under the assumptions of both theorem 3.4 and 3.5 and if inf(P) is finite, strong duality holds that is*

$$\inf(P^2) = \sup(D_L^2) = \sup(D_F^2) = \sup(D_{FL}^2).$$

Proof: Using Theorem 3.4 and Theorem 3.5 we have $\sup(D_L^2) = \sup(D_F^2) = \sup(D_{FL}^2)$. Now it remains to show that $\inf(P^2) = \sup(D_L^2)$. The system of inequalities $F_2(x, r) - \inf(P^2) < 0, G_2(x, r) \leq 0$ has no solution in R^n.

Using Lemma 1.3 to the above system, it follows that there exists $u^* > 0, q^* \in R^m$ such that for each $x, r \in R^n, u^*(F_2(x, r) - \inf(P^2)) + q^{*T} G_2(x, r) \geq 0$.

$u^* \neq 0$, because if $u^* = 0$ then $q^{*T} G_2(x, r) \geq 0$ and by (CQ) we get $x', r' \in R$ such that $q^{*T} G_2(x', r') < 0$ which is a contradiction.

Dividing above inequality by u^*

$$F_2(x, r) - \inf(P^2) + q_0^{*T} G_2(x, r) \geq 0, \quad \text{for all } x, r \in R^n$$

where $q_0^* = q^* / u^*$ and equivalently $\inf_{x, r \in R^n} [F_2(x, r) + q_0^{*T} G_2(x, r)] \geq \inf(P^2)$. Thus $\sup(D_L^2) \geq \inf(P^2)$. Also from Theorem 3.1 $\sup(D_L^2) \leq \inf(P^2)$. From Theorem 3.4 and Theorem 3.5 it can be concluded that,

$$\inf(P^2) = \sup(D_L^2) = \sup(D_F^2) = \sup(D_{FL}^2).$$

4 Conclusions

In this paper the Fenchel-Rockafellar approach is used for the mathematical programming problem with inequality constraints to construct second order duals. Using distinct perturbations of the primal problem Lagrange dual, Fenchel dual and Fenchel Lagrange dual are derived. These duals are different from Wolfe's and Mond , Weir's second order dual in the sense that these duals are unconstrained mathematical programming problems. Using the concept of this paper second order duals for multi objective nonlinear programming problems can be formulated, which is the future research scope of this paper.

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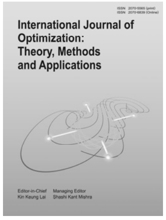

B-invexity and B-monotonicity of Non-differentiable Functions

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Abstract. Several new kinds of generalized B-invexity and generalized invariant B-monotonicity are introduced for non-differentiable functions. The relations among (quasi) B-preinvexity, (pseudo, quasi) B-invexity and invariant (pseudo, quasi) B-monotonicity are studied by using the Clarke's subdifferential of non-differentiable functions and a series of examples. Some new results are obtained, which can be viewed as an extension of some known results.

Keywords: B-preinvexity, B-invexity, invariant B-monotonicity, Clarke's subdifferential, relations.

1 Introduction

Convexity is a common assumption made in mathematical programming. There have been increasing attempts to weaken the convexity of objective functions, see for example [1-11] and references therein. An interested generalization for convexity is B-vexity, which was introduced and studied by Bector and Singh [4]. They studied some properties of B-vex functions in the settings of differentiable and non-differentiable, respectively. Later, B-preinvexity was introduced by Suneja et al. [5] as an extension of preinvexity and B-vexity. At the same time, B-vexity was generalized to pseudo (quasi) B-vexity and (pseodo, quasi) B-invexity in the setting of differential in [6]. Recently, B-vexity was studied by Li et al. [7] in the setting of non-differential and some necessary and sufficient results are obtained by means of the Clarke's subdifferential.

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A concept related to the convexity is the monotonicity of the mappings. In 1990, Karamardian and Schaible [8] studied the relations between the convexity of a real-valued function and the monotonicity of its gradient mapping. Yang et al. [9] and Jabarootian and Zafarani [10] investigated the relations between invexity and generalized invariant monotonicity in the settings of differentiable and non-differentiable, respectively.

Motivated and inspired by works in [7-9], in this paper, we will introduce several new notions of generalized B-invexity and generalized invariant monotonicity , which are called pseudo (quasi) B-invexity and invariant pseudo (quasi) B-monotonicity, and study the relations among (quasi) B-preinvexity, (pseudo, quasi) B-invexity and invariant (pseudo, quasi) B-monotonicity by means of the Clarke's subdifferential of non-differentiable functions and a series of examples. Some new results are obtained, which can be viewed as an extension and improvement of corresponding results in [2,6,7,10].

2 Generalized B-invexity and Generalized Invariant B-monotonicity

Throughout this paper, let X be a real Banach space endowed with a norm $\|\cdot\|$ and dual space X^*. We denote by $2^{X^*}, \langle\cdot,\cdot\rangle, [x^1, x^2]$ and (x^1, x^2) the family of all nonempty subset of X^*, the dual pair between X and X^*, the line segment for $x^1, x^2 \in X$ and the interior of $[x^1, x^2]$, respectively. Let K be a nonempty subset of $X, \eta: K\times K \to X$ a vector valued mapping and $f: X \to R$ a function.

K is said to be an invex set with respect to η (see [1]) if for any $x^1, x^2 \in K$ and any $\lambda \in [0,1]$ one has $x^1 + \lambda\eta(x^2, x^1) \in K$.

From now on, unless otherwise specified, we assume that K is a nonempty invex set with respect to η.

Let f be locally Lipschitz continuous at $x \in X$ and v be an any other vector in X. The Clarke's generalized directional derivative of f at x in the direction v is defined by $f^0(x; v) = \limsup_{y\to x, t\downarrow 0} \dfrac{f(y+tv)-f(y)}{t}$. The Clarke's generalized subdifferential of f at x is defined by $\partial^c f(x) = \{\xi \in X^*: f^0(x;v) \ge \langle \xi, v\rangle, \quad \forall v \in X\}$. As shown in [11], $\partial^c f(x)$ is a nonempty convex set and $f^0(x;v) = \max_{\xi\in\partial^c f(x)} \langle \xi, v\rangle$ for all $v \in X$.

Lemma 2.1 [11] (Mean-value theorem) Let $x^1, x^2 \in X$ and $f: X \to R$ be locally Lipschitz continuous near each point of a nonempty closed convex set containing the line segment $[x^1, x^2]$. Then there exist a point $u \in (x^1, x^2)$ and $\xi \in \partial^c f(u)$ such that $f(x^1) - f(x^2) = \langle \xi, x^1 - x^2\rangle$.

In following, we will introduce the concepts of pseudo (quasi) B-invexity and of invariant pseudo (quasi) B-monotonicity.

Definition 2.1 Let $b: K\times K\times[0,1] \to R_+$ be a function. The function f is said to be

(i)[5,6] B-preinvex on K with respect to η and b if for any $x^1, x^2 \in K$ and any $\lambda \in [0,1]$ one has $f(x^2 + \lambda\eta(x^1, x^2)) \le \lambda b(x^1, x^2, \lambda) f(x^1) + (1-\lambda b(x^1, x^2, \lambda)) f(x^2)$;

(ii) quasi B-preinvex on K with respect to η and b if for any $x^1, x^2 \in K$ and any $\lambda \in [0,1]$ one has $f(x^1) \le f(x^2)$ implies $b(x^1, x^2, \lambda) f(x^2 + \lambda\eta(x^1, x^2)) \le b(x^1, x^2, \lambda) f(x^2)$.

From Definition 2.1, we can easily see that B-preinvexity implies quasi Bpreinvexity with respect to the same η and b. But the converse is not necessarily true, see the following example.

Example 2.1 Let $X = R$ and $K = \left[0, \frac{\pi}{2}\right]$. For any $x \in X$, $x^1, x^2 \in K$, and $\lambda \in [0,1]$, let $\eta(x^1, x^2) = \sin x^1 - \sin x^2$ and

$$f(x) = \begin{cases} x, 0 < x < \frac{\pi}{2}, \\ 1, x = \frac{\pi}{2}, \\ 0, \text{otherewise}, \end{cases}$$

$$b(x^1, x^2, \lambda) = \begin{cases} 0, x^1 = \frac{\pi}{2} \text{ or } x^2 = \frac{\pi}{2} \text{ or } \lambda = 0, \\ 1, \text{otherewise}. \end{cases}$$

We can verify that f is quasi B-preinvex on K with respect to η and b. However, for $x^1 = \frac{\pi}{4}$, and $x^2 = \frac{\pi}{2}$ and $\lambda = \frac{1}{2}$, we have $f(x^2 + \lambda\eta(x^1, x^2)) = \frac{\pi}{2} - \frac{2-\sqrt{2}}{4}$ and $\lambda b(x^1, x^2, \lambda) f(x^1) + (1 - \lambda b(x^1, x^2, \lambda)) f(x^2) = 1$, which indicates that f is not B-preinvex on K with respect to η and b.

Definition 2.2 Let $b : K \times K \to R_+$ be a function. f is said to be

(i) B-invex on K with respect to η and b if for any $x^1, x^2 \in K$ and any $\xi \in \partial^c f(x^2)$ one has $\langle \xi, \eta(x^1, x^2) \rangle \le b(x^1, x^2)(f(x^1) - f(x^2))$;

(ii) quasi B-invex on K with respect to η and b if for any $x^1, x^2 \in K$ and any $\xi \in \partial^c f(x^2)$ one has $f(x^1) \le f(x^2)$ implies $b(x^1, x^2)\langle \xi, \eta(x^1, x^2) \rangle \le 0$;

(iii) pseudo B-invex on K with respect to η and b if for any $x^1, x^2 \in K$ and some $\xi \in \partial^c f(x^2)$ one has $\langle \xi, \eta(x^1, x^2) \rangle \ge 0$ implies $b(x^1, x^2) f(x^1) \ge b(x^1, x^2) f(x^2)$.

From Definition 2.2, we can see that B-invexity implies quasi or pseudo Binvexity with respect to the same η and b. But the converses are not necessarily true, see the following example.

Example 2.2 Let $X = R$ and $K = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For any $x \in X$ and $x^1, x^2 \in K$ let $f(x) = |x|, \eta(x^1, x^2) = \sin x^1 - \sin x^2$ and

$$b(x^1, x^2) = \begin{cases} 1, x^1 x^2 > 0, \\ 0, x^1 x^2 \le 0. \end{cases}$$

We can verify that f is quasi and pseudo B-invex on K with respect to η and b. However, for $x^1 = -\frac{\pi}{4}, x^2 = 0$ and $\xi = -1 \in \partial^c f(x^2)$ it follows that $\langle \xi, \eta(x^1, x^2) \rangle > b(x^1, x^2)(f(x^1) - f(x^2))$, which indicates that f is not B-invex on K with respect to η and b.

Definition 2.3 Let $b : K \times K \to R_+$ be a function and $F : X \to 2^{X^*}$ be a set-valued mapping. F is said to be

(i) invariant B-monotone on K with respect to η and b if for any $x^1, x^2 \in K$ and any $u \in F(x^1), \upsilon \in F(x^2)$ one has $b(x^2, x^1)\langle \upsilon, \eta(x^1, x^2) \rangle + b(x^1, x^2)\langle \upsilon, \eta(x^2, x^1) \rangle \le 0$;

(ii) invariant quasi B-monotone on K with respect to η and b if for any $x^1, x^2 \in K$, some $\upsilon \in F(x^2)$ and any $u \in F(x^1)$ one has $b(x^2,x^1)\langle \upsilon, \eta(x^1,x^2)\rangle > 0$ implies $b(x^1,x^2)\langle \upsilon, \eta(x^2,x^1)\rangle \le 0$;

(iii) invariant pseudo B-monotone on K with respect to η and b if for any $x^1, x^2 \in K$, some $\upsilon \in F(x^2)$ and any $u \in F(x^1)$ one has $b(x^2,x^1)\langle \upsilon, \eta(x^1,x^2)\rangle > 0$ implies $b(x^1,x^2)\langle \upsilon, \eta(x^2,x^1)\rangle \le 0$.

From Definition 2.3, we can see that invariant B-monotonicity implies invariant quasi B-monotonicity and invariant pseudo B-monotonicity implies invariant quasi B-monotonicity. But the converses are not necessarily true, see the following two examples.

Example 2.3 Let $X = R$ and $K = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For any $x \in X$ and $x^1, x^2 \in K$ let $f(x) = |x|, \eta(x^1, x^2) = \sin x^1 - \sin x^2$ and

$$b(x^1,x^2) = \begin{cases} 1, x^2 > 0, \\ 0, x^2 \le 0. \end{cases}$$

We can verify that $\partial^c f$ is invariant quasi B-monotone on K with respect to η and b. However, for $x^1 = 0, x^2 = \frac{\pi}{4}, u = 1 \in \partial^c f(x^1)$ and any $\upsilon \in \partial^c f(x^2)$, due to $b(x^2,x^1)\langle \upsilon, \eta(x^1,x^2)\rangle + b(x^1,x^2)\langle \upsilon, \eta(x^2, x^1)\rangle > 0$, we can conclude that $\partial^c f$ is not invariant B-monotone on K with respect to η and b.

Example 2.4 Let $X = R$ and $K = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For any $x \in X$ and $x^1, x^2 \in K$ let $f(x) = -|x|, \eta(x^1, x^2) = \sin x^1 - \sin x^2$ and

$$b(x^1,x^2) = \begin{cases} 1, x^1 x^2 > 0, \text{ or } x^2 = 0, \\ 0, \text{ otherewise.} \end{cases}$$

We can verify that $\partial^c f$ is invariant quasi B-monotone on K with respect to η and b. However, for $x^1 = \frac{\pi}{4}, x^2 = 0, u = -1 \in \partial^c f(x^1)$ and any $\upsilon \in \partial^c f(x^2)$, we can conclude that $b(x^2,x^1)\langle \upsilon, \eta(x^1,x^2)\rangle \ge 0$ implies $b(x^1,x^2)\langle u, \eta(x^2,x^1)\rangle > 0$. Hence, $\partial^c f$ is not invariant pseudo B-monotone on K with respect to η and b.

3 Relations between (Quasi) B-preinvexity and (Pseudo, Quasi) B-invexity

In this section, we mainly study the relations between (quasi) B-preinvexity and (pseudo, quasi) B-invexity for a locally Lipschitz continuous function $f : X \to R$. For this purpose, we need the following assumptions, which are taken from [9].

Assumption A $f(x^2 + \eta(x^1,x^2)) \le f(x^1), \forall x^1, x^2 \in K$.

Assumption C For any $x^1, x^2 \in K$ and any $\lambda \in [0,1]$, one has $\eta(x^2, x^2 + \lambda\eta(x^1, x^2)) = -\lambda\eta(x^1, x^2)$ and $\eta(x^1, x^2 + \lambda\eta(x^1,x^2)) = (1-\lambda)\eta(x^1,x^2)$.

Yang et al. [9] showed that if η satisfies Assumption C, then $\eta(x^2 + \lambda\eta(x^1, x^2), x^2) = \lambda\eta(x^1, x^2)$ for all $x^1, x^2 \in K$ and $\lambda \in [0,1]$.

Theorem 3.1 Let

(i) $b : K \times K \times [0,1] \to R_+$ be such that $b(x^1, x^2, \cdot)$ is continuous at 0^+ for any fixed $x^1, x^2 \in K$;

(ii) η and b be continuous with respect to the second argument, respectively;

(iii) $\bar{b}$ be bounded, where $\bar{b}(x^1, x^2) = \lim_{\lambda \downarrow 0} b(x^1, x^2, \lambda)$ for all $x^1, x^2 \in K$.

If f is B-preinvex on K with respect to η and b, then f is B-invex on K with respect to η and $\bar{b}$. But the converse is not necessarily true.

Proof: For any given $x^1, x^2 \in K$ and $\varepsilon > 0$, let L be the local Lipschitz constant of f at x^2. Then there exists a constant $0 < \delta < \frac{\varepsilon}{2L}$ such that $|f(x^2) - f(x)| < \frac{\varepsilon}{2}$ and $\|\eta(x^1, x^2) - \eta(x^1, x)\| < \frac{\varepsilon}{2L}$ for all $x \in K$ with $\|x^2 - x\| < \delta$. Consequently, for a small enough number $\lambda > 0$, one has

$$\begin{aligned}
&\frac{f(x + \lambda\eta(x^1, x^2)) - f(x)}{\lambda} \\
&\leq \frac{f(x + \lambda\eta(x^1, x)) - f(x)}{\lambda} + L\|\eta(x^1, x^2) - \eta(x^1, x)\| \\
&\leq \frac{\lambda b(x^1, x, \lambda) f(x^1) + (1 - \lambda b(x^1, x, \lambda)) f(x) - f(x)}{\lambda} + L\|\eta(x^1, x^2) - \eta(x^1, x)\| \\
&\leq b(x^1, x, \lambda)(f(x^1) - f(x^2)) + \frac{\varepsilon}{2}(b(x^1, x, \lambda) + 1).
\end{aligned}$$

Taking the limit as $\lambda \downarrow 0, \varepsilon \downarrow 0$ and $x \to x^2$, since $\bar{b}$ is bounded, we get $\langle \xi, \eta(x^1, x^2) \rangle \leq f^0(x^2; \eta(x^1, x^2)) \leq \bar{b}(x^1, x^2)(f(x^1) - f(x^2))$ for all $\xi \in \partial^c f(x^2)$, which shows that f is B-invex on K with respect to η and $\bar{b}$.

The following example shows that the converse is not true.

Example 3.1 Let $X = R$ and $K = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For any $x \in X, x^1, x^2 \in K$ and $\lambda \in [0,1]$, let $\eta(x^1, x^2) = \frac{x^1 - x^2}{3}$ and

$$f(x) = \begin{cases} 3x, x \geq 0, \\ x, x < 0, \end{cases}$$

$$b(x^1, x^2, \lambda) = \begin{cases} 1, x^2 = 0, x^1 > 0, \\ \lambda, x^2 = 0, x^1 \leq 0, \\ \frac{1}{3}, x^2 \neq 0. \end{cases}$$

Then

$$\bar{b}(x^1, x^2) = \begin{cases} 1, x^2 = 0, x^1 > 0, \\ 0, x^2 = 0, x^1 \leq 0, \\ \frac{1}{3}, x^2 \neq 0. \end{cases}$$

We can verify that f is B-invex on K with respect to η and $\bar{b}$. However, for $x^1 = -\frac{\pi}{4}, x^2 = 0$ and $\lambda = \frac{1}{2}$, we can deduce that $f(x^2 + \lambda\eta(x^1, x^2)) >= \lambda b(x^1, x^2, \lambda) f(x^1) + (1 - \lambda b(x^1, x^2, \lambda)) f(x^2)$. Hence, *f* is not B-preinvex on *K* with respect to η and $\bar{b}$.

Theorem 3.2 Let $b: K\times K \to R_{++}$. If *f* is B-invex on *K* with respect to η and *b* and satisfies Assumption C, then *f* is B-preinvex on *K* with respect to η and $\bar{b}$, where

$$\bar{b}(x^1,x^2,\lambda)=\frac{b(x^1,x^2+\lambda\eta(x^1,x^2))}{\lambda b(x^1,x^2+\lambda\eta(x^1,x^2))+(1-\lambda)b(x^2,x^2+\lambda\eta(x^1,x^2))}$$

for all $x^1,x^2\in K$ *and* $\lambda\in[0,1]$.

Proof: Take arbitrarily $x^1, x^2 \in K$ and $\lambda\in[0,1]$ and let $x^0=x^2+\lambda\eta(x^1,x^2)$. By the definition of B-invexity, for any $\xi\in\partial^c f(x^0)$, we have

$$\left\langle \xi,\eta(x^1,x^0)\right\rangle \le b(x^1,x^0)(f(x^1)-f(x^0)), \tag{1}$$

$$\left\langle \xi,\eta(x^2,x^0)\right\rangle \le b(x^2,x^0)(f(x^2)-f(x^0)). \tag{2}$$

Multiplying (1) by λ and (2) by (1 − λ) and adding them, by Assumption C, we can deduce that $\lambda b(x^1,x^0)f(x^1)+(1-\lambda)b(x^2,x^0)f(x^2)\ge(\lambda b(x^1,x^0)+(1-\lambda)b(x^2,x^0))f(x^0)$, which implies that $f(x^2+\lambda\eta(x^1,x^2))\le\lambda\bar{b}(x^1,x^2,\lambda)f(x^1)+(1-\lambda\bar{b}(x^1,x^2,\lambda)f(x^2)$, where

$$\bar{b}(x^1,x^2,\lambda)=\frac{b(x^1,x^2+\lambda\eta(x^1,x^2))}{\lambda b(x^1,x^2+\lambda\eta(x^1,x^2))+(1-\lambda)b(x^2,x^2+\lambda\eta(x^1,x^2))}.$$

Therefore, the assertion of the theorem holds.

The following two examples show that there are not direct implications between quasi B-invexity and quasi B-preinvexity.

Example 3.2 Let $X=R$ and $K=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. For any $x\in X, x^1,x^2\in K$ and $\lambda\in[0,1]$ let $f(x)=-|x|$ and

$$\eta(x^1,x^2)=\begin{cases}\sin x^1-\sin x^2, & x^1x^2\ge 0,\\ 0, & x^1x^2<0,\end{cases}$$

$$b(x^1,x^2,\lambda)=\begin{cases}1, x^1x^2\ge 0,\\ \lambda, x^1x^2<0.\end{cases}$$

Then

$$\bar{b}(x^1,x^2)=\lim_{\lambda\downarrow 0} b(x^1,x^2,\lambda)=\begin{cases}1, x^1x^2\ge 0,\\ 0, x^1x^2<0.\end{cases}$$

We can verify that *f* is quasi B-preinvex on *K* with respect to η and *b*. For $x^1=-\frac{\pi}{4}$, $x^2=0$ and $\xi=-1\in\partial^c f(x^2)$, we can deduce that $f(x^1)\le f(x^2)$ implies $\bar{b}(x^1,x^2)\left\langle\xi,\eta(x^1,x^2)\right\rangle>0$. Hence, *f* is not quasi B-invex on *K* with respect to η and $\bar{b}$.

Example 3.3 $X = R$ and $K = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For any $x \in X, x^1, x^2 \in K$ and $\lambda \in [0,1]$ let $f(x) = -|x|$, $\eta(x^1, x^2) = \sin x^1 - \sin x^2$ and

$$b(x^1, x^2, \lambda) = \begin{cases} 1, x^1 x^2 > 0, \\ 0, x^1 x^2 \le 0. \end{cases}$$

Then

$$\bar{b}(x^1, x^2) = \lim_{\lambda \downarrow 0} b(x^1, x^2, \lambda) = \begin{cases} 1, x^1 x^2 > 0, \\ 0, x^1 x^2 \le 0. \end{cases}$$

We can verify that f is quasi B-invex on K with respect to η and $\bar{b}$. However, for $x^1 = \frac{\pi}{4}, x^2 = -\frac{\pi}{6}$ and $\lambda = \frac{1}{2}$, since $f(x^1) \le f(x^2)$ implies $b(x^1, x^2, \lambda) f(x^2 + \lambda\eta(x^1, x^2)) > b(x^1, x^2, \lambda) f(x^2)$, we can conclude that f is not quasi B-preinvex on K with respect to η and b.

Theorem 3.3 Let
(i) $b: K \times K \times [0,1] \to R_+$ be such that $b(x^1, x^2, \cdot)$ is continuous at 0^+ for any fixed $x^1, x^2 \in K$;
(ii) η and b be continuous with respect to the second argument, respectively;
(iii) $\bar{b}$ be bounded, where $\bar{b}(x^2, x^1) = \lim_{x \to x^1, \lambda \downarrow 0} b(x^2, x, \lambda)$ for all $x^1, x^2 \in K$.
If f is quasi B-preinvex on K with respect to η and b, then for any $\xi \in \partial^c f(x^1)$ we have

$$f(x^2) < f(x^1) \Rightarrow \bar{b}(x^2, x^1)\langle \xi, \eta(x^2, x^1) \rangle \le 0.$$

Proof: Let $x^1, x^2 \in K$ with $f(x^2) < f(x^1)$ and $L > 0$ be the local Lipschitz constant of f at x^1. By the continuity of f at x^1, we know that there exists a constant $\delta > 0$ such that $f(x^2) < f(x)$ for all $x \in X$ with $\|x - x^1\| < \delta$, For any $\lambda \in [0,1]$ and any $x \in K$ with $\|x - x^1\| < \delta$, by the quasi B-preinvexity of f, we have $b(x^2, x, \lambda) f(x + \lambda\eta(x^2, x) \le b(x^2, x, \lambda) f(x)$. Consequently,

$$\begin{aligned} & b(x^2, x, \lambda) \frac{f(x + \lambda\eta(x^2, x^1)) - f(x)}{\lambda} \\ & \le b(x^2, x, \lambda)(\frac{f(x + \lambda\eta(x^2, x)) - f(x)}{\lambda} + L\|\eta(x^2, x) - \eta(x^2, x^1)\|) \\ & \le b(x^2, x, \lambda) L \|\eta(x^2, x) - \eta(x^2, x^1)\|. \end{aligned}$$

Taking the limit for the last inequality as $x \to x^1$ and $\lambda \downarrow 0$, since b is bounded, we get $\bar{b}(x^2, x^1)$ $f^0(x^1; \eta(x^2, x^1)) \le 0$ and then $\bar{b}(x^2, x^1)\langle \xi, \eta(x^2, x^1) \rangle \le 0, \forall \xi \in \partial^c f(x^1)$, which indicates that the assertion of the theorem holds.

Theorem 3.4 Let $b: K \times K \to R_+$ be continuous with respect to the second argument. If f is quasi B-invex on K with respect to η and b and satisfies Assumption C, then f is quasi B-preinvex on K with respect to η and $\bar{b}$, where $\bar{b}(x^1, x^2, \lambda) = b(x^1, x^2 + \lambda\eta(x^1, x^2)) b(x^2, x^2 + \lambda\eta(x^1, x^2))$ for all x^1, $x^2 \in K$ and $\lambda \in [0,1]$.

Proof: Take arbitrarily $x^1, x^2 \in K$ and let $f(x^1) \le f(x^2)$. In order to show that f is quasi B-preinvex on K, it suffices to prove that the set

$$\Omega=\{x^2+\lambda\eta(x^1,x^2):\bar{b}(x^1,x^2,\lambda)f(x^2+\lambda\eta(x^1,x^2))>\bar{b}(x^1,x^2,\lambda)f(x^2),\lambda\in[0,1]\}$$

is empty. It is evident that Ω is equivalent to the set

$$\Omega'=\{x^2+\lambda\eta(x^1,x^2):f(x^2+\lambda\eta(x^1,x^2))>f(x^2),\bar{b}(x^1,x^2,\lambda)>0,\lambda\in[0,1]\}.$$

Assume to the contrary that $\Omega'\neq\emptyset$. By the continuity of f, we know that the set

$$\Omega''=\{x^2+\lambda\eta(x^1,x^2):f(x^2+\lambda\eta(x^1,x^2))>f(x^2),\bar{b}(x^1,x^2,\lambda)>0,\lambda\in[0,1]\}$$

is nonempty. Consequently, for every $\bar{x}\in\Omega''$, there exist $\bar{\lambda}\in(0,1)$ such that $\bar{x}=x^2+\bar{\lambda}\eta(x^1,x^2)$, $\bar{b}(x^1,x^2,\bar{\lambda})>0$ and $f(\bar{x})>f(x^2)\geq f(x^1)$. For any $\xi\in\partial^c f(\bar{x})$, by the quasi B-invexity of f, it follows that $b(x^2,\bar{x})\langle\xi,\eta(x^2,\bar{x})\leq 0$ and $b(x^1,\bar{x})\langle\xi,\eta(x^1,\bar{x})\leq 0$, which together with Assumption C shows that $b(x^1,\bar{x})b(x^2,\bar{x})\langle\xi,\eta(x^1,x^2)\rangle=0$ and then

$$\langle\xi,\eta(x^1,x^2)\rangle=0,\ \forall\bar{x}\in\Omega'',\ \forall\xi\in\partial^c f(\bar{x}).\tag{3}$$

Again by the continuity of f and b, we can find $\lambda^*\in(0,1)$ with $\lambda^*<\bar{\lambda}$ such that $f(x^2+\lambda^*\eta(x^1,x^2))=f(x^2)$ and $f(x^2+\lambda\eta(x^1,x^2))\in\Omega''$ for all $\lambda\in(\lambda^*,\bar{\lambda}]$. By Lemma 2.1 and (3), there exist $\tilde{\lambda}\in(\lambda^*,\bar{\lambda})$ and $\gamma\in\partial^c f(x^2+\tilde{\lambda}\eta(x^1,x^2))$ such that $0<f(x^2+\bar{\lambda}\eta(x^1,x^2))-f(x^2)=(\bar{\lambda}-\lambda^*)\langle\gamma,\eta(x^1,x^2)\rangle=0$, which is a contradiction. Hence, the assertion of the theorem holds.

4 Relations between Generalized B-invexity and Generalized Invariant B-monotonicity

In this section, we mainly study the relations between (pseudo, quasi) B-invexity of a locally Lipschitz continuous function f and invariant (pseudo, quasi) B-monotonicity of its subdifferential mapping $\partial^c f$.

The following result is a direct consequence of Definition 2.1.

Theorem 4.1 Let $b:K\times K\to R_+$ If f is B-invex on K with respect to η and b, then $\partial^c f$ is invariant B-monotone on K with respect to the same η and b.

Theorem 4.2 Let b : $K\times K\to R_{++}$. If $\partial^c f$ is invariant B-monotone on K with respect to η and b and satisfies Assumptions A and C, then there exists a function $\lambda: K\times K\to(0,1)$ such that f is B-invex on K with respect to η and $\bar{b}$, $\bar{b}(x^2,x^1)=\dfrac{b(x^1+\lambda(x^1,x^2)\eta(x^2,x^1),x^1)}{b(x^1,x^1+\lambda(x^1,x^2)\eta(x^2,x^1))}$ for all $x^1,x^2\in K$.

Proof: Let $\partial^c f$ be invariant B-monotone on K. For any $x^1,x^2\in K$. by Assumption A and Lemma 2.1, there exist a constant related to x^1,x^2 in (0, 1),denoted by $\lambda(x^1,x^2)$, and a point $\xi\in\partial^c f(x^0)$ such that $f(x^1)-f(x^2)\leq f(x^1)-f(x^1+\eta(x^2,x^1))=-\langle\xi,\eta(x^2,x^1)\rangle$, where $x^0=x^1+\lambda(x^1,x^2)\eta(x^2,x^1)$. By Assumption C, for any $\upsilon\in\partial^c f(x^1)$, it follows that $b(x^0,x^1)\langle\xi,\eta(x^2,x^1)\rangle\geq b(x^1,x^0)\langle\upsilon,\eta(x^2,x^1)\rangle$ and then

$$b(x^0,x^1)(f(x^2)-f(x^1))\geq b(x^1,x^0)\langle\upsilon,\eta(x^2,x^1)\rangle$$

which is equivalent to $\frac{b(x^0,x^1)}{b(x^1,x^0)}(f(x^2)-f(x^1)) \geq \left\langle \upsilon, \eta(x^2,x^1) \right\rangle$. The last inequality shows that f is B-invex on K with respect to η and $\bar{b}$.
The following result is a direct consequence of Definition 2.2.

Theorem 4.3 Let $b : K \times K \to R_+$. If f is quasi B-invex on K with respect to η and b, then $\partial^c f$ is invariant quasi B-monotone on K with respect to η and $\bar{b}$, where $\bar{b}(x^1,x^2)=b(x^2,x^1)$ for all $x^1,x^2 \in K$. But the converse is not necessarily true for same η and b.

Example 4.1 Let X, K, f, η and b be the same as in Example 2.4. Then $\partial^c f$ is invariant quasi B-monotone on K with respect to η and b. However, for $x^1=-\frac{\pi}{4}, x^2=0$ and $\xi=-1\in \partial^c f(x^2)$, we can deduce that $f(x^1)\leq f(x^2)$ implies $b(x^1,x^2)\langle \xi, \eta(x^1,x^2)\rangle > 0$ This shows that f is not quasi B-invex on K with respect to η and b.

Theorem 4.4 Let b : $K \times K \to R_+$. If $\partial^c f$ is invariant pseudo B-monotone on K with respect to η and b and satisfies Assumptions A and C, then there exists a function $\lambda : K \times K \to (0, 1)$ such that f is pseudo B-invex on K with respect to η and $\bar{b}$, where $\bar{b}(x^1,x^2)=b(x^2+\lambda(x^1,x^2)\eta(x^1,x^2),x^2)$ for all $x^1,x^2 \in K$.

Proof: Take arbitrarily $x^1,x^2 \in K$. For x^2 and $x^2+\eta(x^1,x^2)$, by Lemma 2.1, there exist a constant related to x^1,x^2 in (0, 1), denoted by $\lambda(x^1,x^2)$, and a point $u\in \partial^c f(x^2+\lambda(x^1,x^2)\eta(x^1,x^2))$ such that

$$f(x^2+\eta(x^1,x^2))-f(x^2)=\langle u,\eta(x^1,x^2)\rangle, \tag{4}$$

Assume to the contrary that the assertion of the theorem is not true. Then there exist $x^1,x^2 \in K$ such that

$$\langle \upsilon,\eta(x^1,x^2)\rangle \geq 0, \forall \upsilon\in \partial^c f(x^2), \tag{5}$$

and $\bar{b}(x^1,x^2)f(x^1)<\bar{b}(x^1,x^2)f(x^2)$, which shows that $f(x^1)<f(x^2)$. By Assumption A and (4), we have $b(x^2+\lambda(x^1,x^2)\eta(x^1,x^2),x^2)\langle u,\eta(x^1,x^2)\rangle<0$. By Assumption C, we get $b(x^2+\lambda(x^1,x^2)$ $\eta(x^1,x^2),x^2)\left\langle u,\eta(x^2,x^2+\lambda(x^1,x^2)\eta(x^1,x^2))\right\rangle > 0$ By the invariant pseudo B-monotonicity of $\partial^c f$, for some $\omega\in \partial^c f(x^2)$, we obtain

$$b(x^2,x^2+\lambda(x^1,x^2)\eta(x^1,x^2))\left\langle \omega,\eta(x^2+\lambda(x^1,x^2)\eta(x^1,x^2),x^2)\right\rangle<0$$

which implies that $\langle \omega,\eta(x^1,x^2)\rangle<0$. This contradicts (5). Therefore, the assertion of the theorem holds.

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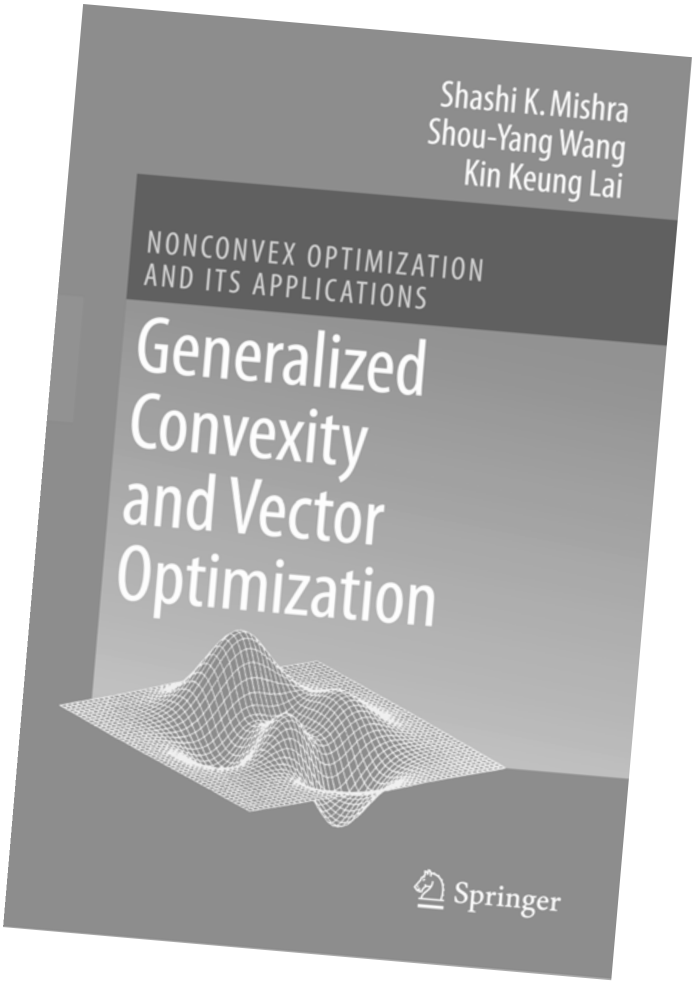

About this book

The present book discusses the Kuhn-Tucker Optimality, Karush-Kuhn-Tucker Necessary and Sufficient Optimality Conditions in presence of various types of generalized convexity assumptions. Wolfe-type Duality, Mond-Weir type Duality, Mixed type Duality for Multiobjective optimization problems such as Nonlinear programming problems, Fractional programming problems, Nonsmooth programming problems, Nondifferentiable programming problems, Variational and Control problems under various types of generalized convexity assumptions.

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V-INVEX FUNCTIONS AND VECTOR OPTIMIZATION summarizes and synthesizes an aspect of research work that has been done in the area of Generalized Convexity over the past several decades. Specifically, the book focuses on V-invex functions in vector optimization that have grown out of the work of Jeyakumar and Mond in the 1990's. V-invex functions are areas in which there has been much interest because it allows researchers and practitioners to address and provide better solutions to problems that are nonlinear, multi-objective, fractional, and continuous in nature. Hence, V-invex functions have permitted work on a whole new class of vector optimization applications.

There has been considerable work on vector optimization by some highly distinguished researchers including Kuhn, Tucker, Geoffrion, Mangasarian, Von Neuman, Schaiible, Ziemba, etc. The authors have integrated this related research into their book and demonstrate the wide context from which the area has grown and continues to grow. The result is a well-synthesized, accessible, and usable treatment for students, researchers, and practitioners in the areas of OR, optimization, applied mathematics, engineering, and their work relating to a wide range of problems which include financial institutions, logistics, transportation, traffic management, etc.

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