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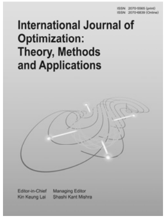

Unifying Efficiency and Weak Efficiency in Generalized Quasiconvex Vector Minimization on the Real-line[♣]

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Abstract. Usually the concepts of *efficient* and *weakly efficient* solution are associated to a multiobjective optimization problem. Both notions may be described in terms of a *preference relation* determined by a closed convex cone with nonempty interior - typically the nonnegative orthant of a finite dimensional space. We present a unified approach for dealing with both notions at the same time under generalized quasiconvexity assumptions on the objectives defined on the real-line. Since most algorithms in scalar minimization involve the solvability of a one-dimensional problem to find the next iterate, it is expected that our results be applied in the vector case.

We established several characterizations of the nonemptiness of the solution set, and also various characterizations when besides boundedness is required. To that end we used a notion of relaxed convexity for vector functions introduced ealier by one of the authors.

Keywords. nonconvex vector optimization, quasiconvex vector functions, efficiency, weak efficiency, generalized convexity.

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1 Introduction and Formulation of the Problem

In multiobjective optimization several criterion functions must be minimized simultaneously. Very often, no single point minimizing all criteria at once may be found, and so the concept of optimality has to be modified. Usually, the notion of *efficient* or *weakly efficient* solution is considered. A point is called efficient or Pareto-optimal, if there does not exist a different point with smaller or equal objective function values, such that there is a decrease in at least one objective function value; a point is called weakly efficient or weakly Pareto-optimal, if there exists no other point with strictly smaller objective function value. Both notions may be described in terms of a *preference relation* determined by a closed convex cone with nonempty interior - typically the nonnegative orthant of some $\mathbb{R}^m$.

More precisely, given a nonempty set $K \subseteq \mathbb{R}^n$, a closed convex cone $P \subseteq \mathbb{R}^m$, and a vector-valued function, $F : K \to \mathbb{R}^m$, a point $\overline{x} \in K$ is an efficient solution to the problem

$$\min\{F(x) : x \in K\} \tag{1}$$

if

$$F(x) - F(\overline{x}) \notin -P \setminus l(P) \quad \forall\, x \in K, \tag{2}$$

where $l(P) = P \bigcap (-P)$; whereas when $\operatorname{int} P \neq \emptyset$, $\overline{x}$ is a weakly efficient solution to (1) if

$$F(x) - F(\overline{x}) \notin -\operatorname{int} P \quad \forall\, x \in K. \tag{3}$$

We denote by $E = E(K)$ the set of efficient solutions to (1), and by $E_W = E_W(K)$ the set of weakly efficient solutions. Since $P \subseteq \mathbb{R}^m \setminus -(P \setminus l(P)) \subseteq \mathbb{R}^m \setminus (-\operatorname{int} P)$, every efficient solution is also weakly efficient. We refer to [24, 25, 19] for a theoretical treatment of even more general vector optimization problems.

Problem (3) has been extensively studied mainly when $P = \mathbb{R}^m_+$; among the recent publications for the existence of weakly efficient solutions, we quote [7, 13, 15, 16, 17, 24, 27, 9] and references therein; whereas results about existence of efficient solutions may be found in [10, 12, 19, 25].
The case $K \subseteq \mathbb{R}$ is of particular interest since it is known that most algorithms in scalar minimization (e.g. Newton, gradient, projection methods) require to solve a one-dimensional minimization problem. It is expected that similar devices be applied in multiobjective optimization. Thus, purpose of this work is to give a unified approach for both problems in the particular case when $K \subseteq \mathbb{R}$ under generalized quasiconvexity assumptions, so we will be interested in the problem:

$$\text{find } \overline{x} \in K : F(x) - F(\overline{x}) \in S \quad \forall\, x \in K, \tag{4}$$

where S is a cone satisfying $S + P \subseteq S$. This inclusion is satisfied if either $S = \mathbb{R}^m \setminus -(P \setminus l(P))$ or $S = \mathbb{R}^m \setminus (-\operatorname{int} P)$.

Although in recent years nonlinear-scalarizations schemes are being employeed in multiobjetive optimization problems-specially in the absence of the standar convexity assumptions ([11])-the linear-scalarization tool, or weighting method, is among the main procedures, even in the quasiconvexity case, [11, 21, 17, 18, 23]. However, the main drawback lies on the choice of the parameters since it is not known in advance which ones give rise to solutions. In fact, bad choices of these parameters can lead to unbounded scalar optimization problems, or under convexity, this method computes only *proper* efficient solutions [21].

The notion of semistrict (S)-quasiconvexity for vector functions, suitable for studying Problem (4), is recalled in Section 2, where it is also established some relationship with other classes of functions. Section 3 provides several characterizations of the nonemptiness of the solution set to (4), and also various necessary and sufficient conditions when in addition boundedness is required. Several examples are presented in each section showing our results are in some sense optimal.

We end this section by recalling the definition of asymptotic cone. Given a set C, the asymptotic cone of C, denoted by C^∞, is defined by

$$C^\infty \doteq \left\{u : \exists\, t_k \downarrow 0, \exists\, x^k \in C, t_k x^k \to u\right\},$$

which is a closed cone. The "recession" term instead of asymptotic is employed when convex sets are considered. Some of the main properties of asymptotic cones may be found in [26, Chapter 3].

Proposition 1.1 The following assertions hold.

(a) $K_1 \subseteq K_2$ implies $K_1^\infty \subseteq K_2^\infty$;
(b) let $K \subseteq \mathbb{R}^n$, then K is bounded if and only if $K^\infty = \{0\}$;
(c) Let $\{K_i\}$, $i \in I$, be any family of nonempty subsets of X, then

$$\left(\bigcap_{i\in I} K_i\right)^\infty \subseteq \bigcap_{i\in I} (K_i)^\infty .$$

If, in addition, $\bigcap_i K_i \neq \emptyset$ and each set $K_i, i \in I,$ is closed and convex, then we obtain an equality in the previous inclusion.

2 Semistrict (*S*)-quasiconvexity and Related Properties

Let X, Y be real normed vector spaces. We are also given a nonempty set $S \subseteq Y,$ a nonempty convex set $K \subseteq X$, and a mapping $F : K \to Y$. It is requested to find

$$\bar{x} \in K : \ F(y) - F(\bar{x}) \in S \quad \forall\, y \in K. \tag{5}$$

A point $\bar{x} \in K$ satisfying (5) is called a (global) *S-minimal* of F (on K) and the set of such points is denoted by E_S. Obviously $E_S \neq \emptyset$ implies that $0 \in S$.
In connection to problem (5) the following definition introduced in [15], and further developed in [16, 17], will play an important role. In the following: "$\mathrm{co}(A)$" stands for the convex hull of the set A, which is the smallest convex set containing A; given $x \neq y$, we set

$$[x, y] \doteq \{tx + (1-t)y : 0 \le t \le 1\}.$$

Similarly for $[x, y[,]x, y[$.

Definition 2.1. The vector function $F : K \to Y$ is said to be:

(i) ([15, 16]) semistrictly (S)-quasiconvex at $y \in K$, if for every $x \in K, x \neq y$,

$$F(x) - F(y) \in -S \Rightarrow F(\xi) - F(y) \in -S \quad \forall\, \xi \in]x, y[.$$

We say that F is semistrictly (S)-quasiconvex (on K) if it is at every $y \in K$.

(ii) ([15]) explicitly (S)-quasiconvex at $y \in K$, if it is semistrictly (S)-quasiconvex and semistrictly $(Y \setminus -S)$-quasiconvex at y.

We say that F is *explicitly* (S)-*quasiconvex* (on K) if it is at every $y \in K$.

When $Y = \mathbb{R}$, the previous definition reduces to quasiconvexity of real-valued functions in case $S \doteq \mathbb{R}_+ = [0, +\infty)$, and to semistrict quasiconvexity if $S \doteq \mathbb{R}_{++} = (0, +\infty)$.

In what follows, $P \subseteq Y$ is a (not necessarily pointed) convex cone, $P \neq Y$. The set $P^* \subseteq Y^*$, with Y^* being the topological dual of Y, is the (nonnegative) polar cone defined by

$$P^* = \{p^* \in Y^* : \langle p^*, p \rangle \geq 0 \ \ \forall\, p \in P\},$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between Y and Y^*.

We recall two important facts if additionally P is also closed: the first one is the bipolar theorem $(P^{**} = P)$,

$$p \in P \Longleftrightarrow \langle p^*, p \rangle \geq 0 \ \ \forall\, p^* \in P^*, \tag{6}$$

and in case $\operatorname{int} P \neq \emptyset$, we have

$$p \in \operatorname{int} P \Longleftrightarrow \langle p^*, p \rangle > 0 \ \ \forall\, p^* \in P^* \setminus \{0\}. \tag{7}$$

Some generalizations of convexity in connection with optimality conditions are provided in [5, 6, 22].

The next definition collects some of the main notions of generalized convexity employed in the existence theory of vector optimization problems appearing in the literature.

Definition 2.2 Let $K \subseteq X$ be a convex set and P be a convex cone. The function $F : K \to Y$, is said to be

(i) P-convex if for all $x, y \in K$,

$$\alpha F(x) + (1-\alpha) F(y) \in F(\alpha x + (1-\alpha) y) + P \quad \text{for all } \alpha \in (0,1).$$

In particular, F is $\mathbb{R}^m_+$-convex if and only if each component of F is convex;

(ii) properly P-quasiconvex ([12]) if for each $x, y \in K$,

$$F([x, y]) \subseteq \{F(x), F(y)\} - P,$$

or equivalently, the set

$$\{\xi \in K : F(\xi) \notin \lambda + P\} \text{ is convex for all } \lambda \in \mathrm{Y}.$$

(iii) naturally P-quasiconvex ([28]), if for each $x, y \in K$,

$$F([x,y]) \subseteq [F(x), F(y)] - P.$$

(iv) scalarly P-quasiconvex ([23]) if

$$x \in K \mapsto \langle p^*, F(x) \rangle \text{ is quasiconvex for every } p^* \in P^*.$$

(v) P-quasiconvex ([12, 24]) if the set

$$\{\xi \in K : F(\xi) \in \lambda - P\} \text{ is convex for all } \lambda \in \mathrm{Y}.$$

In particular, F is $\mathbb{R}^m_+$-quasiconvex if and only if each component of F is quasiconvex;

(vi) (int $P \neq \emptyset$) [15, 16, 17] *semistrictly* $(Y \setminus -\text{int } P)$*-quasiconvex* (see Definition 2.1, [15]), if for every $x, y \in K$,

$$F(x) - F(y) \notin \text{int } P \Rightarrow F(\alpha x + (1-\alpha)y) - F(y) \notin \text{int } P \,\forall\, \alpha \in (0,1).$$

(vii) *semistrictly* $(Y \setminus -(P \setminus l(P)))$*-quasiconvex* (see Definition 2.1) if for every $x, y \in K$,

$$F(x) - F(y) \notin P \setminus l(P) \Rightarrow F(\alpha x + (1-\alpha)y) - F(y) \notin P \setminus l(P) \,\forall\, \alpha \in (0,1).$$

In general, as noted in [12, 15], there is no relationship between the notions of P-convexity and proper P-quasiconvexity.

Also with no further assumption on P, the class of naturally P-quasiconvex functions is strictly larger than the preceding two classes of functions as shown in [28]. On the other hand, the class of P-quasiconvex functions is strictly larger than the scalarly P-quasiconvex functions as shown in [23] (where the terminology $*$-quasiconvexity is applied), and this latter class of functions is equivalent to the class of naturally P-quasiconvex functions as established in Theorem 2.3 below.

Furthermore, the function $F(x) = (\frac{1}{1+|x|^2}, |x|), x \in \mathbb{R}$, is semistrictly $(\mathbb{R}^2 \setminus -\text{int } \mathbb{R}^2_+)$-quasiconvex but not $\mathbb{R}^2_+$-quasiconvex.

We also notice that there is no relationship between (*vi*) and *(vii)*, and also between (*v*) and (*vii*). Indeed, by taking $K = [0, 1]$, the function $F = (f_1, f_2)$ given by

$$f_1(x) = \begin{cases} -x+1 & \text{if } \; 0 < x \le 1 \\ 0 & \text{if } \; x = 0 \end{cases}, \quad f_2(x) = x,$$

is semistrictly $(\mathbb{R}^2 \setminus -(\mathbb{R}^2_+ \setminus \{0\}))$-quasiconvex without being semistrictly $(\mathbb{R}^2 \setminus -\text{int } \mathbb{R}^2_+)$-quasiconvex (and therefore neither $\mathbb{R}^2_+$-quasiconvex, because of Theorem 2.3 below) since $F(1) - F(0) = (0,1) \notin \text{int } \mathbb{R}^2_+$ and $F(1/2) - F(0) = (1/2, 1/2) \in \text{int } \mathbb{R}^2_+$; whereas the function

$$f_1(x) = x, \quad f_2(x) = \begin{cases} 0 & \text{if } \; 0 \ge 1 \\ 1 & \text{if } \; 0 \le x < 1, \end{cases} \qquad K = [0, +\infty),$$

is $\mathbb{R}^2_+$-quasiconvex (thus semistrictly $(\mathbb{R}^2 \setminus -\text{int } \mathbb{R}^2_+)$-quasiconvex by the next theorem), but not semistrictly $(\mathbb{R}^2 \setminus -(\mathbb{R}^2_+ \setminus \{0\}))$-quasiconvex at 0 since $F(1) - F(0) = (1, -1) \notin \mathbb{R}^2_+ \setminus \{0\}$ and $F(1/2) - F(0) = (1/2, 0) \in \mathbb{R}^2_+ \setminus \{0\}$.

Certainly, proper P-quasiconvexity and P-quasiconvexity are the more common generalizations of the notion of quasiconvexity for real-valued functions. The semistrict $(Y\setminus -\text{int }P)$-quasiconvexity and the semistrict $(Y\setminus -(P\setminus l(P)))$-quasiconvexity provide further generalizations as well: the first being suitable when dealing with weak efficiency ([15, 16, 17]) and the second for efficiency.

The next theorem establishes the relationship between the notions introduced in Definition 2.2.

Theorem 2.3 Assume $K\subseteq X$ is a convex set and $P\subseteq Y$ a convex cone such that $P\neq Y$. Concerning Definition def-conv-gen, we have the following assertions: itemize

(a) $(i)\Rightarrow(iii)\Rightarrow(v)$, $(ii)\Rightarrow(iii)\Rightarrow(iv)$, $(iii)\Rightarrow(vi)$, $(iii)\Rightarrow(vii)$;
(b) if in addition P is closed then

$$(iii)\Longleftrightarrow(iv)\Rightarrow(v)\Rightarrow(vi).$$

(c) if additionally $P\bigcup(-P)=Y$ then,

$$(v)\Longleftrightarrow(ii)\Longleftrightarrow(iii)\text{ and }(v)\Rightarrow(iv);$$

(d) if additionally $P\bigcup(-P)=Y$ and P is closed (and so $\text{int }P\neq\emptyset$), then

$$(ii)\Longleftrightarrow(iii)\Longleftrightarrow(iv)\Longleftrightarrow(v)\Longleftrightarrow(vi)\Longleftrightarrow(vii).$$

Proof: (a): The implication $(iii)\Rightarrow(vii)$ follows from Proposition 2.6 below with $S=Y\setminus(-P\setminus l(P))$; the other implications appear in Theorem 2.6 of [15] since they are still valid in general normed vector spaces.
(b): When $\text{int }P\neq\emptyset$, the equivalence is proved in Proposition 3.9 of [18]. However, the proof still remains valid in the general case with obvious changes. We reproduce the proof for reader's convenience.

Assume that F satisfies (iv). We need to check that given $t\in R$ and $x^*\in P^*$, the set $K_t=\{z\in K:\langle x^*,F(z)\rangle\leq t\}$ is convex. Indeed, if $x,y\in K_t$ then by natural-P-quasiconvexity of F, for all $z\in[x,y]$ there exists $\lambda\in[0,1]$ and $u\in P$ such that $F(z)=\lambda F(x)+(1-\lambda)F(y)-u$. Hence,

$$\langle x^*,F(z)\rangle=\lambda\langle x^*,F(x)\rangle+(1-\lambda)\langle x^*,F(y)\rangle-\langle x^*,u\rangle\leq t$$

thus $z\in K_t$, so K_t is convex.
Conversely, assume that F is not naturally-P-quasiconvex. Then, there exist $x,y\in K$ and $z\in\,]x,y[$ such that for all $\mu\in[0,1]$, $F(z)\notin\mu F(x)+(1-\mu)F(y)-P$. Thus, by (6), for every $\mu\in[0,1]$ there exists $x^*\in P^*\setminus\{0\}$ such that

$$\langle x^*,F(z)\rangle>\langle x^*,\mu F(x)+(1-\mu)F(y)\rangle.$$

We can suppose that $\|x^*\|=1$. By Alaoglu's theorem (see [1] or [Theorem 5.25][2]), the set $B=\{y^*\in Y^*:\|y^*\|\leq 1\}$ is weak-star compact. Thus,

$$x^*\in B_0\doteq P^*\bigcap B=\{y^*\in P^*:\|y^*\|\leq 1\}$$

is also weak-star compact. Setting $f(y^*,\mu)=\langle y^*, F(z)-\mu F(x)-(1-\mu)F(y)\rangle$, we can apply the Sion minimax theorem (see for instance [23]) to get

$$\max_{y^*\in B_0}\min_{\mu\in[0,1]} f(y^*,\mu)=\min_{\mu\in[0,1]}\max_{y^*\in B_0} f(y^*,\mu)>0.$$

Hence there exists $x^*\in B_0$ such that

$$\langle x^*, F(z)\rangle > \mu\langle x^*, F(x)\rangle+(1-\mu)\langle x^*, F(y)\rangle \quad \forall\, \mu\in[0,1].$$

In particular, we get $x^*\in P^*$ and $\langle x^*, F(z)\rangle > \langle x^*, F(x)\rangle$ and $\langle x^*, F(z)\rangle > \langle x^*, F(y)\rangle$.
Thus F is not scalarly P-quasiconvex.
For the other implications we refer Theorem 2.6 of [15].

(c): It may be found in Theorem 2.6 of [15].

(d): This is a consequence of (*a*), (*b*) and (*c*) and the fact that $P=Y\setminus -\text{int}\, P$ and $P\setminus l(P)=\text{int}\, P$. Notice that in this case there exists $p^*\in P^*\setminus\{0\}$ such that $P=\{p\in Y:\langle p^*, p\rangle\geq 0\}$ (see, for instance, [15]).

For a given $y\in K$, we set

$$S_y \doteq \{x\in K: F(x)-F(y)\in -S\}.$$

The proof of the next lemma follows immediately from Definition 2.1.

Lemma 2.4 [16] Given *S*, *K* as above and $F:K\to Y$. For fixed $y\in K$, the following two assertions are equivalent: itemize

(a) F is semistrictly (S)-quasiconvex at y;
(b) $[x,y]\subseteq S_y$ for all $x\in S_y$.

We recall that *C* is starshaped at $y\in C$, if $[x,y]\subseteq C$, for all $x\in C$.
Thus, if *C* is starshaped at $y\in C$, then

$$C^\infty=\bigcap_{t>0} t(C-y). \tag{8}$$

In case $0\in S$, we have $y\in S_y$ for all $y\in K$, and therefore, we obtain the following theorem without any continuity assumption.

Theorem 2.5 Given *S*, *K* as above with $0\in S$, and $F:K\to Y$. The following assertions hold.

(a) F is semistrictly (S)-quasiconvex at $y\in K$ if, and only if, S_y is starshaped at y;
(b) Assume, in addition that K is closed. If F is semistrictly (S)-quasiconvex at y then

$$(S_y)^\infty=\{v\in K^\infty: F(y+\lambda v)-F(y)\in -S \quad \forall\, \lambda>0\}.$$

Proof: (*a*) follows from the previous lemma. Part (*b*) is a consequence of (8), since

$$v\in (S_y)^\infty \Longleftrightarrow y+tv\in S_y \quad \forall\, t>0 \Longleftrightarrow v\in K^\infty,\ F(y+tv)-F(y)\in -S \quad \forall\, t>0.$$

Proposition 2.6 Let $\emptyset\neq S\subseteq Y$ such that $tS\subseteq S$ for all $t>0$; $K\subseteq X$ be convex; $P\subseteq Y$ be a convex cone satisfying $S+P\subseteq S$. Let $F:K\to Y$ be given.

(a) If F is P-convex, then F is explicitly (S)-quasiconvex.
(b) Assume that $0 \in S$ and that F is naturally P-quasiconvex, then F is semistrictly (S)-quasiconvex.

Proof: (a): Let $x, y \in K$ and $x \neq y$. By P-convexity we have $F(tx+(1-t)y) \in tF(x)+(1-t)F(y) -P$ for all $t \in (0,1)$. Thus, by setting $\xi_t \doteq tx+(1-t)y$, we have $F(\xi_t)-F(y) \in t(F(x)-F(y))-P$ for all $t \in (0,1)$. If $F(x)-F(y) \in -S$, then $F(\xi_t)-F(y) \in -tS-P \subseteq -S$, proving the semistrict (S)-quasiconvexity. On the other hand, if $F(x)-F(y) \in Y \setminus S$ then $F(\xi_t)-F(y) \in t(Y\setminus S)-P \subseteq Y \setminus S$. It proves the semistrict $(Y \setminus -S)$-quasiconvexity, which completes the proof.

(b): Given $x, y \in K$, we have that for all $\xi \in]x, y[$,

$$F(\xi) \in [F(x), F(y)] - P.$$

Thus, for every $\xi \in]x, y[$ there exists $\mu \in [0,1]$ such that $F(\xi)-F(y) \in \mu(F(x)-F(y))-P$. If $F(x)-F(y) \in -S$, then $F(\xi)-F(y) \in -\mu S - P \subseteq -S-P \subseteq -S$, proving the desired result.

The class of explicit (S)-quasiconvexity also includes that of explicit quasiconvexity componentwise when $P=\mathbb{R}^m_+$ and $S = Y \setminus (-\text{int } \mathbb{R}^m_+)$, $S = Y \setminus -(\mathbb{R}^m_+ \setminus \{0\})$. More generally, we say that $F: K \to Y$ is *explicitly scalarly P-quasiconvex* if it is scalarly P-quasiconvex (see (iv) in Definition 2.2) and semistrictly scalarly P-quasiconvex: the latter means, that

$$x \in K \mapsto \langle p^*, F(x) \rangle \text{ is semistrictly quasiconvex for every } p^* \in P^*.$$

Proposition 2.7 Let $\emptyset \neq K \subseteq X$ be convex and closed. Assume that $P \subseteq Y$ is a closed, convex cone. If $F: K \to Y$ is explicitly scalarly P-quasiconvex then it is explicitly $(P \setminus l(P))$-quasiconvex, and explicitly (int P)-quasiconvex (here int $P \neq \emptyset$).

Proof: Let $x, y \in K$. If $F(x)-F(y) \in -(P \setminus l(P))$, then $\langle p^*, F(x)-F(y) \rangle \leq 0$ for all $p^* \in P^*$, and $\langle q^*, F(x)-F(y) \rangle < 0$ for some $q^* \in P^*$. By hypothesis, $\langle p^*, F(\xi)-F(y) \rangle \leq 0$ and $\langle q^*, F(\xi)-F(y) \rangle < 0$ for all $\xi \in]x, y[$. Hence, $F(\xi)-F(y) \in -P \setminus l(P)$ for all $\xi \in]x, y[$.
Assume that $F(x)-F(y) \notin (P \setminus l(P))$. If on the contrary, $F(\xi)-F(y) \in P \setminus l(P)$ for some $\xi \in]x, y[$, then $\langle p^*, F(\xi)-F(y) \rangle \geq 0$ for all $p^* \in P^*$ and $\langle q^*, F(\xi)-F(y) \rangle > 0$ for some $q^* \in P^*$. Thus, $\langle p^*, F(x)-F(y) \rangle \geq 0$ for all $p^* \in P^*$ and $\langle q^*, F(x)-F(y) \rangle > 0$. Hence, $F(x)-F(y) \in P \setminus l(P)$, a contradiction. This completes the proof that F is explicitly $(P \setminus l(P))$-quasiconvex.
The explicit (int P)-quasiconvexity of F follows from Theorem 5.3 of [15].

When P^* is the weak-star closed convex hull of its extreme directions, one obtains part of the previous proposition under weaker assumptions. In what follows, extrd P^* stands for the set of extreme directions of P^*: here $q^* \in$ extrd P^* if and only if $q^* \in P^* \setminus \{0\}$ and for all $q_1^*, q_2^* \in P^*$ such that $q^* = q_1^* + q_2^*$ we actually have $q_1^*, q_2^* \in \mathbb{R}_{++} q^*$.

Proposition 2.8 Let $\emptyset \neq K \subseteq X$ be convex and closed, $P \neq Y$ be a closed convex cone in Y and $F: K \to Y$ be given.

(a) Assume that P^* is the weak-star closed convex hull of extrd P^*. If for all $p^* \in$ extrd P^*,

$$x \in K \mapsto \langle p^*, F(x) \rangle \text{ is quasiconvex and semistrictly quasiconvex,}$$

then it is explicitly $(P\setminus l(P))$-quasiconvex. If, in addition P^* is polyhedral with int $P\neq\emptyset$, then F is also explicitly (int P)-quasiconvex.

(b) Let $D\doteq\{p^*\in P^*:\|p^*\|=1\}$.

(b1) If for all $p^*\in D$,

$$x\in K\mapsto\langle p^*,F(x)\rangle \text{ is quasiconvex and semistrictly quasiconvex,}$$

then F is explicitly $(P\setminus l(P))$-quasiconvex.

(b2) (int $P\neq\emptyset$) If for all $p^*\in D$,

$$x\in K\mapsto h_{p^*}(x)\doteq\langle p^*,F(x)\rangle \text{ is quasiconvex,}$$

then F is semistrictly $(Y\setminus -\text{int } P)$-quasiconvex.

(b3) If for all $p^*\in D, h_{p^*}$ is semistrictly quasiconvex, then F is semistrictly (int P)-quasiconvex.

Proof: (a): It follows the same argument of the previous proposition and using the following two facts (see (6)):

$$p\in P\Longleftrightarrow\langle p^*,p\rangle\geq 0\ \ \forall\ p^*\in \text{extrd } P^*, \tag{9}$$

and if for some $z\in Y$ there is $p^*\in P^*$ such that $\langle p^*,z\rangle<0$, then there is $q^*\in$ extrd P^* satisfying $\langle p^*,z\rangle<0$. The last assertion follows easily.

We apply a similar reasoning as that in the preceding proposition to prove Parts (b1), (b2) and (b3).

Other suficient conditions ensuring explicit (int P)-quasiconvexity may be found in [15].

Remark 2.9 Conditions ensuring that P^* is the weak-star closed convex hull of extrd P^*, may be found in Remark 2.2 of [3]. It is true, in particular, when int $P\neq\emptyset$. In that paper was also proved that P-quasiconvexity is equivalent to the quasiconvexity of

$$x\in K\mapsto\langle p^*,F(x)\rangle \text{ for every } p^*\in \text{extrd } P^*.$$

When $P=\mathbb{R}^m_+$, extrd $\mathbb{R}^m_+$ reduces to the canonical basis of $\mathbb{R}^m$. Thus, for any vector function the assumptions in the previous theorem read by saying that each of its components is quasiconvex and semistrictly quasiconvex.

Remark 2.10 By virtue of Remark 2.9, (b2) of Proposition 2.8 remains true if D is replaced by extrd P^*; whereas (b3) continues to be valid as above, under the stronger assumption $P^*=$ co(extrd P^*), which is satisfied if P is polyhedra or if P^* is a kind of ice-cream cone.

Example 2.11 This example shows that explicit $(\mathbb{R}^m_+\setminus\{0\})$-semistrict quasiconvexity does not imply the semistrict quasiconvexity of each component. Take $K=[0,+\infty), P=\mathbb{R}^2_+$ and $F=(f_1,f_2)$ with

$$f_1(x)=x,\quad f_2(x)=\begin{cases}0 & \text{if}\quad x=0,\\ 1 & \text{if}\quad x\neq 0.\end{cases}$$

Here, $E=\{0\}$ and $E_W=\{0\}$.

3 Unifying Efficiency and Weak Efficiency on the Real-line: the Main Results

We are concerned in the problem

$$\text{find } \overline{x} \in K : F(y) - F(\overline{x}) \in S \quad \forall\, y \in K. \tag{10}$$

The following hypothesis will be needed in the sequel.

Hypothesis (A): Problem (10) admits a solution for all compact convex set $K \subseteq X$.
Concerning the validity of Hypothesis (A), we have the next lemma.

Lemma 3.1 Let X, Y be normed vector spaces, $K \subseteq X$ compact, $P \subseteq Y$ be convex cone such that $P \neq Y$, and $F : K \to Y$ be given. Then, Hypothesis (A) holds under any of the following circumstances:

(a) ([19, Theorem 5.1]) $S = Y \setminus (-P \setminus l(P))$ and the set $\{x \in K : F(x) - F(y) \in -P\}$ is closed for all $y \in K$;

(b) ([16, Theorem 3.2]) $S = Y \setminus -\text{int}\, P$, $\text{int}\, P \neq \emptyset$ and the set $\{x \in K : F(x) - F(y) \notin \text{int}\, P\}$ is closed for all $y \in K$.

It is known from [4] (see also [14,24]) that if F is P-lower semicontinuous then (a) and (b) hold. We recall that ([24]) $F : K \to Y$ is *P-lower semicontinuous* (P-lsc) at $x_0 \in K$ if for any open set $V \subseteq Y$ such that $F(x_0) \in V$ there exists an open neighborhood $U \subseteq X$ of x_0 such that $F(U \cap K) \subseteq V + P$. We shall say that F is P-lsc (on K) if it is at every $x_0 \in K$.
We point out that $F = (f_1, \cdots, f_m)$ is $\mathbb{R}^m_+$-lsc if and only if each f_i is lsc.

In connection to Problem (4) and following the same line of reasoning as in [13,14,15], we set

$$R_s \doteq \bigcap_{y \in K} \left\{ v \in K^\infty : F(y + \lambda v) - F(y) \in -S \quad \forall\, \lambda > 0 \right\}. \tag{11}$$

From now on, we restrict ourselves to the case $X = \mathbb{R}$. Notice that P-lower semicontinuity of F is not assumed in the next two theorems.

Theorem 3.2 Let $\emptyset \neq K \subseteq X$ be closed and convex, let $\emptyset \neq S \subseteq Y$ such that $0 \in S$, and $F : K \to Y$ be semistrictly (S)-quasiconvex. Assume that $E_S \neq \emptyset$, then

$$(E_S)^\infty \subseteq R_S. \tag{12}$$

If, in addition, every S_y is closed and convex for all $y \in K$, then $(E_S)^\infty = R_S$.

Proof: If $E_S \neq \emptyset$ then

$$\emptyset \neq (E_S)^\infty = (\bigcap_{y \in K} S_y)^\infty \subseteq \bigcap_{y \in K} (S_y)^\infty = R_S,$$

by Theorem 2.5. The last part is a consequence of the previous inclusion by taking into account Proposition 1.1.

The closedness of S_y is satisfied if F is P-lsc; and the convexity of each S_y holds if either $X = \mathbb{R}$ or $Y = \mathbb{R}$ (in such a case $P = \mathbb{R}_+$) provided F is semistrictly (S)-quasiconvex. However, when $X = \mathbb{R}$, the same result is obtained without the closedness of S_y, as Theorem 3.4 below shows.

We now exhibit some instances where the inclusion in (12) does not hold without the semistrict (*S*)-quasiconvexity assumption. The first example concerns efficiency and the second weak efficiency.

Example 3.3 We really need the semistrict (*S*)-quasiconvexity in Theorem 3.2.

(i) Take $K=[0,+\infty)$, $S=\mathbb{R}^2\setminus-(\mathbb{R}^2_+\setminus\{0\})$ and

$$f_1(x)=|x-1|,\quad f_2(x)=\begin{cases}4 & \text{if } x\in[0,2],\\ 6-x & \text{if } x\in(2,+\infty).\end{cases}$$

An easy computation shows that $E=E_S=\{1\}\bigcup(2,+\infty)$, $(E_S)^\infty=[0,+\infty)$, whereas $R_S=\{0\}$. Setting $F=(f_1,f_2)$, we notice that $F(3)-F(3/2)=(3/2,-1)\notin\mathbb{R}^2_+\setminus\{0\}$ but $F(2)-F(3/2)=(1/2,0)$ $\in\mathbb{R}^2_+\setminus\{0\}$.

(ii) Take $K=\mathbb{R}$, $S=(\mathbb{R}^2\setminus-int\ \mathbb{R}^2_+)$ and

$$f_1(x)=\begin{cases}|x| & \text{if } x\in(-\infty,2],\\ |x-4| & \text{if } x\in(2,+\infty).\end{cases}\qquad f_2(x)=\begin{cases}4 & \text{if } x\in(-\infty,0],\\ 4+x & \text{if } x\in[0,2),\\ |x-3|+5 & \text{if } x\in(2,+\infty).\end{cases}$$

Here $E_W=E_S=(-\infty,0]\bigcup\{4\}$, $(E_S)^\infty=(-\infty,0]$ and $R_S=\{0\}$. Setting $F=(f_1,f_2)$, we observe that $F(0)-F(3)=(-1,-1)\notin\text{int}\ \mathbb{R}^2_+$ but $F(2)-F(3)=(1,1)\in\text{int}\ \mathbb{R}^2_+$.

Theorem 3.4 Let $\emptyset\neq K\subseteq\mathbb{R}$ be closed and convex, let $\emptyset\neq S\subseteq Y$ such that $0\in S$, and $F:K\to Y$ be semistrictly (*S*)-quasiconvex. Assume that $E_S\neq\emptyset$, then E_S is convex and $(E_S)^\infty=R_S$.

Proof: By virtue of the first part of the previous theorem, we need only to prove that $R_S\subseteq(E_S)^\infty$. Take $0\neq v\in R_S$. We consider $v>0$ (the case $v<0$ is analyzed in a similar way). Take any $x\in E_S$, we will check that $x+\lambda v\in E_S$ for all $\lambda>0$. Let $y\in K$ and $\lambda>0$. If $x+\lambda v\in\text{co}(\{x,y\})$, then $F(x+\lambda v)-F(y)\in-S$ by assumption on *F*. If $x+\lambda v\notin\text{co}(\{x,y\})$, we distinguish the cases $x<y$ and $y<x$. If $x<y$, then $x+\lambda v\in(y,y+\lambda v)$. Therefore $F(y)-F(x+\lambda v)\in S$ since $F(y+\lambda v)-F(y)$ $\in-S$. If $y<x(<x+\lambda v)$, then by writing $x+\lambda v=y+\rho v$ for some $\rho>0$, we obtain $F(x+\lambda v)-F(y)=F(y+\rho v)-F(y)\in-S$. Hence $x+\lambda v\in E_S$ for all $\lambda>0$, which implies that $v\in(E_S)^\infty$, and thus the proof is completed.

One can find examples in higher dimension where the convexity of E_W fails under the standar convexity assumption on *F*, see for instance [13].

Theorem 3.5 Let $\emptyset\neq K\subseteq\mathbb{R}$ be a closed convex set, let $S\subseteq Y$ be given. Assume that $F:K\to Y$ is semistrictly (*S*)-quasiconvex. The following assertions are equivalent:

(a) $R_S=\{0\}$;

(b) $\exists\, r>0,\ \forall\, x\in K\setminus K_r, \exists\, y\in K_r: F(y)-F(x)\notin S$, where $K_r=K\bigcap[-r,r]\neq\emptyset$.

Proof: $(a)\Rightarrow(b)$: Suppose to the contrary that for all $n\in\mathbb{N}$ there exists $x_n\in K\setminus K_n$, $F(y)-F(x_n)$ $\in S$ for all $y\in K_n$. Then, for $y\in K$ fixed and $n>|y|$, we have $F(y)-F(x_n)\in S$ for all $n>|y|$. Assume that $\frac{x_n}{|x_n|}\to v\in K^\infty$. We consider only the case $v=1$ (the other is entirely similar). Let $\lambda>0$

and choice n satisfying $(|x_n|>)n > \max\{|y+\lambda|, |y|\}$. As $y+\lambda \in (y, x_n)$ and $F(y)-F(x_n) \in S$, we obtain $F(y+\lambda)-F(y) \in -S$. Thus, $1 = v \in R_S$, a contradiction.

$(b) \Rightarrow (a)$: Take any $x \in K_r$ and assume that $0 \neq v \in R_S$. Then, for some $\lambda_0 > 0$, $x+\lambda_0 v \in K \setminus K_r$. Thus, by ($b$), there is $y \in K_r \subseteq K$ such that $F(y)-F(x+\lambda_0 v) \notin S$. On the other hand $x+\lambda_0 v = y + \lambda_1 v$ for some $\lambda_1 > 0$ by convexity of $K \subseteq \mathbb{R}$. Therefore $F(y+\lambda_1 v)-F(y) \notin -S$, contradicting the choice of $v \in R_S$.

The implication $(a) \Rightarrow (b)$ in the preceding theorem remains valid if K is a subset of a finite dimensional space under the additional assumption of P-lower semicontinuity on F, as proven in [16].

By numerical aspects one might be interested in knowing, a priori, when the solution set is bounded since, like in scalar optimization, convergence of minimizing sequences depends heavily on that property. In this spirit, next theorem plays a important role. The assumption on the complementary of S in the next theorem is satisfied when

$$S = Y \setminus -(P \setminus l(P)),\ S = Y \setminus -\text{int } P,$$

with P being a (not necessarily pointed) convex cone.

Theorem 3.6 Let $\emptyset \neq K \subseteq \mathbb{R}$ be a closed convex set and $0 \in S \subseteq Y$ be a cone satisfying $(Y \setminus S) + (Y \setminus S) \subseteq Y \setminus S$. Assume that $F : K \to Y$ is semistrictly (S)-quasiconvex and Hypothesis (A) is satisfied. The following assertions are equivalent:

(a) $R_S = \{0\}$;
(b) $\exists\, r > 0,\ \forall\, x \in K \setminus K_r, \exists\, y \in K_r : F(y)-F(x) \notin S$, where $K_r = K \bigcap [-r, r] \neq \emptyset$;
(c) E_S is nonempty and bounded.

Proof: $(a) \Longleftrightarrow (b)$: It follows from the previous result.

$(b) \Rightarrow (c)$: For every $n \in \mathbb{N}$, set $K_n = \{x \in K : |x| \leq n\}$. We may suppose, without loss of generality $K_n \neq \emptyset$ for all $n \in \mathbb{N}$. Let us consider the problem

$$\text{find } \overline{x} \in K_n : F(y)-F(\overline{x}) \in S \quad \forall\, y \in K_n. \tag{13}$$

By Hypothesis (A), it admits a solution, say $x_n \in K_n$ for all $n \in \mathbb{N}$. We choose $n \in \mathbb{N}$, such that $n \leq r < n+1$. We claim that x_{n+1} is also a solution to problem (10). In fact, suppose there exists $y \in K$ with $|y| > n+1$ satisfying $F(y)-F(x_{n+1}) \notin S$. By assumption, there exists $y_r \in K_r \subseteq K_{n+1}$: $F(y_r)-F(y) \notin S$. Thus,

$$F(y_r)-F(x_{n+1}) = F(y_r)-F(y)+F(y)-F(x_{n+1}) \in (Y \setminus S)+(Y \setminus S) \subseteq Y \setminus S,$$

a contradiction.

$(c) \Rightarrow (a)$: It is a consequence of Theorem 3.4.

Example 3.7 The semistrict (S)-quasiconvexity of F cannot be deleted in the preceding theorem. Take $K = [0, +\infty)$ and $F = (f_1, f_2)$ with

$$f_1(x) = e^{-x}, \quad f_2(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq 1, \\ e^{-x^2+1} & \text{if} \quad x \geq 1. \end{cases}$$

Here $E=\{0\}$, $E_W=\{0\}$. We obtain $F(0)-F(2)=(1-e^{-2},-e^{-3})\notin \mathbb{R}^2_+\setminus\{0\}$ but $F(1)-F(2)=(e^{-1}-e^{-2}, 1-e^{-3})\in \text{int}\,\mathbb{R}^2_+$. Hence, F is neither semistrictly $(\mathbb{R}^2\setminus-(\text{int}\,\mathbb{R}^2_+))$-quasiconvex at 2 nor semistrictly $(\mathbb{R}^2\setminus(-\mathbb{R}^2_+\setminus\{0\}))$-quasiconvex at 2. Here (b) does not hold for either $S=\mathbb{R}^2\setminus(-\text{int}\,\mathbb{R}^2_+)$ or $S=\mathbb{R}^2\setminus(-\mathbb{R}^2_+\setminus\{0\})$.

The preceding theorem has several useful consequences which are expressed in the two following corollaries. The next result valid for quasiconvex real functions will be used subsequently. It is taken from [8], see also [20].

Lemma 3.8. If f is quasiconvex on $\mathbb{R}$, then there is $a\in[-\infty,+\infty]$ such that f is non-increasing on $(-\infty,a]$ and non-decreasing on the rest (with the convention $(-\infty,-\infty]=[+\infty,+\infty)=\emptyset$)

Corollary 3.9. Let $\emptyset\neq K\subseteq\mathbb{R}$ be convex and closed, $P\neq Y$ be a closed convex cone in Y, and $F:K\to Y$ be given.

(a) Set $D=\{p^*\in P^*:\|p^*\|=1\}$. Assume that for all $p^*\in D$, $x\in K\mapsto\langle p^*,F(x)\rangle$ is lsc.
 $(a1)$ If F is semistrictly $(Y\setminus(-P\setminus l(P)))$-quasiconvex then,

$$E\neq\emptyset \text{ and bounded}\Rightarrow \text{argmin}_K\langle q^*,F(\cdot)\rangle\neq\emptyset\ \ \forall\, q^*\in D.$$

 $(a2)$ If for all $p^*\in D$, $x\in K\mapsto\langle p^*,F(x)\rangle$ is semistrictly quasiconvex, then,

$$E\neq\emptyset \text{ and } E\neq\mathbb{R}\ \Rightarrow\exists\, q^*\in D, \text{argmin}_K\langle q^*,F(\cdot)\rangle\neq\emptyset.$$

(b) Assume that P^* is the weak-star closed convex hull of extre P^* (see the previous section), and that for all $p^*\in\text{extrd}\,P^*$, $x\in K\mapsto\langle p^*,F(x)\rangle$ is lsc.
 $(b1)$ If F is semistrictly $(Y\setminus(-P\setminus l(P)))$-quasiconvex then,

$$E\neq\emptyset \text{ and bounded}\Rightarrow \text{argmin}_K\langle q^*,F(\cdot)\rangle\neq\emptyset\ \ \forall\, q^*\in \text{extrd}\,P^*.$$

 $(b2)$ If for all $p^*\in\text{extrd}\,P^*$, $x\in K\mapsto\langle p^*,F(x)\rangle$ is semistrictly quasiconvex then,

$$E\neq\emptyset \text{ and } E\neq\mathbb{R}\ \Rightarrow\exists\, q^*\in \text{extrd}\,P^*, \text{argmin}_K\langle q^*,F(\cdot)\rangle\neq\emptyset.$$

Proof: We only prove (b) ((a) is similar). We first observe that Hypothesis (A) is satisfied: one can check that $\{x\in K:F(x)-F(y)\in -P\}$ is closed for all $y\in K$ by virtue of (9). Thus, the result is a consequence of Theorem 3.6 since its Part (b) reads

$$\exists\, r>0,\ \forall\, x\in K\setminus K_r, \exists\, y\in K_r: F(y)-F(x)\in -P\setminus l(P).$$

This implies, in particular, that (see again (9))

$$\exists\, r>0,\ \forall\, x\in K\setminus K_r, \exists\, y\in K_r, \forall\, q^*\in\text{extrd}\,P^*, \langle q^*,F(y)\rangle\leq\langle q^*,F(x)\rangle.$$

This inequality along with the lower semicontinuity of $\langle q^*,F(\cdot)\rangle$ imply that $\text{argmin}_K\langle q^*,F(\cdot)\rangle\neq\emptyset$.

(b): We know that lower semicontinuity and semistrict quasiconvexity imply quasiconvexity. Assume, to the contrary, that $\text{argmin}_K\langle q^*,F(\cdot)\rangle=\emptyset$ for all $q^*\in\text{extrd}\,P^*$. Then, by the explicit quasiconvexity, either $\langle q^*, F(\cdot)\rangle$ is non-increasing on K for all $q^* \in \text{extrd } P^*$, or non-decreasing on K for all $q^* \in \text{extrd } P^*$. Let us consider $K = [a, +\infty)$ (resp. $K = (-\infty, a]$) then for all $q^* \in \text{extrd } P^*$, $\langle q^*, F(\cdot)\rangle$ is strictly decreasing (resp. strictly increasing). We claim that $E = \emptyset$. In fact, let $\overline{x} \in K$ and take $x > \overline{x}$, $x \in K$. Then

$$\langle q^*, F(x) - F(\overline{x})\rangle < 0 \ \ \forall\, q^* \in \text{extrd } P^*.$$

It turns out that

$$\langle q^*, F(x) - F(\overline{x})\rangle \le 0 \ \ \forall\, q^* \in P^*.$$

Both inequalities imply

$$F(x) - F(\overline{x}) \in -P \setminus l(P),$$

showing that $\overline{x} \notin E$, proving the claim.
We consider the case $K = \mathbb{R}$. Since $E \neq \emptyset$, it is not difficult to prove the existence of $p^*, q^* \in \text{extrd } P^*$ such that $\langle p^*, F(\cdot)\rangle$ is strictly increasing and $\langle q^*, F(\cdot)\rangle$ is strictly decreasing. This allows us to prove that $E = \mathbb{R}$, a contradiction.

We recall that if P is a closed convex cone with $\text{int } P \neq \emptyset$, then P^* is the weak-star closed convex hull of $\text{extrd } P^*$ by Remark 2.2 in [3].

Corollary 3.10. Let $\emptyset \neq K \subseteq \mathbb{R}$ be convex and closed, $P \neq Y$ be a closed convex cone in Y such that $\text{int } P \neq \emptyset$, and $F : K \to Y$ be given.

(a) If F is semistrictly $(Y \setminus -\text{int } P)$-quasiconvex and P-lsc, then

$$E_W \neq \emptyset \text{ and bounded} \Rightarrow \text{argmin}_K \langle p^*, F(\cdot)\rangle \neq \emptyset \text{ and compact } \ \forall\, p^* \in D,$$

where $D \doteq \{p^* \in P^* : \| p^* \| = 1\}$.

(b) Assume that F is P-lower semicontinuous on K and

$(b1)$ $\text{argmin}_K \langle q, F(\cdot)\rangle \neq \emptyset$ for all $q \in D$, with D as in (a). If for all $q \in D$, the function $\langle q, F(\cdot)\rangle$ is semistrictly quasiconvex, then

$$E_W = \overline{\text{co}}\left(\bigcup_{q \in D} \text{argmin}_K \langle q, F(\cdot)\rangle \right).$$

$(b2)$ $\text{argmin}_K \langle q, F(\cdot)\rangle \neq \emptyset$ for all $q \in \text{extrd } P^*$. If for all $q \in \text{extrd } P^*$, the function $\langle q, F(\cdot)\rangle$ is semistrictly quasiconvex and $P^* = \text{co}(\text{extrd } P^*)$ then

$$E_W = \overline{\text{co}}\left(\bigcup_{q \in \text{extrd } P^*} \text{argmin}_K \langle q, F(\cdot)\rangle \right).$$

(c) Assume that $\text{extrd } P^*$ is finite (i.e., P^* polyhedra) and

$(c1)$ that for all $p^* \in \text{extrd } P^*$,

$$x \in K \mapsto \langle p^*, F(x)\rangle \text{ is quasiconvex and lsc. Then}$$

$$E_W \neq \emptyset \text{ and } E_W \neq \mathbb{R} \Rightarrow \exists\, q^* \in \text{extrd } P^*,\ \text{argmin}_K \langle q^*, F(\cdot)\rangle \neq \emptyset.$$

($c2$) that for all $p^* \in$ extrd P^*,

$$x \in K \mapsto \langle p^*, F(x)\rangle \text{ is semistrictly quasiconvex and lsc. Then}$$

$$E_W \neq \emptyset \text{ and compact} \Longleftrightarrow \text{argmin}_K \langle q^*, F(\cdot)\rangle \neq \emptyset \text{ compact } \forall\, q^* \in \text{extrd } P^*.$$

Proof: (a): Clearly Hypothesis (A) holds since P-lsc implies (actually equivalent to) the closedness of $\{x \in K : F(x) - F(y) \notin \text{int } P\}$ for all $y \in K$, see [4] (see also [14]). From (b) of Theorem 3.6 it follows that

$$\exists\, r > 0,\ \forall\, x \in K \setminus K_r, \exists\, y \in K_r : F(y) - F(x) \in -\text{int } P.$$

Taking into account (7), the previous relation becomes

$$\exists\, r > 0,\ \forall\, x \in K \setminus K_r, \exists\, y \in K_r : \langle p^*, F(y)\rangle < \langle p^*, F(x)\rangle\ \forall\, p^* \in P^* \setminus \{0\}.$$

This inequality along with the (see Theorem 5.5 in [24, Chapter 1]) lower semicontinuity of $\langle p^*, F(\cdot)\rangle$ implies that $\text{argmin}_K \langle p^*, F(\cdot)\rangle \neq \emptyset$ and compact for all $p^* \in D$.

($b1$) (($b2$) is similar): As in (a), for all $q \in D$, $\langle q, F(\cdot)\rangle$ is lsc. Thus, $\text{argmin}_K \langle q, F(\cdot)\rangle$ is closed for all $q \in D$. Since every lsc and semistrictly quasiconvex real-valued function is quasiconvex, we conclude that F is semistrictly $(Y \setminus -\text{int } P)$-quasiconvex (see Proposition 2.8). By Theorem 3.4, E_W is convex. The closedness of E_W follows from the P-lower semicontinuity of F, since it is equivalent to the closedness of $\{x \in K : F(x) - F(y) \notin \text{int } P\}$ for all $y \in K$ (see [4] or [14]), and E_W is the intersection of such sets. Therefore

$$A_0 \doteq \overline{\text{co}}\Big(\bigcup_{q \in D} \text{argmin}_K \langle q, F(\cdot)\rangle\Big) \subseteq E_W.$$

Set $h_q(x) = \langle q, F(\cdot)\rangle$. Thus, A_0 is of the form $(-\infty, +\infty)$, $[\alpha, +\infty)$, $(-\infty, \alpha]$, $[\alpha, \beta]$ for some $-\infty < \alpha \leq \beta < +\infty$. Obviously, in the first case there is nothing to prove. We only consider the case $A_0 = [\alpha, \infty)$. If $x \in E_W \setminus A_0$, by (7) we may choose $q \in D$ such that $h_q(x) \leq h_q(\alpha)$. Take any $x_q \in$ argmin h_q; then $h_q(x_q) < h_q(x)$ since $x \notin A_0$, and therefore $h_q(\alpha) < h_q(x)$ reaching a contradiction.

(cl) follows from a similar reasoning as in (b) of the previous corollary; ($c2$) is a consequence of (a) and ($b2$).

In the case when P^* is a polyhedral cone, that is,

$$P^* = \bigcup_{t \geq 0} t\ \text{co}(\{a_1, \cdots, a_m\}),$$

it is known that (Corollary 5.6 [24, Chapter 1]) F is P-lsc if and only if $\langle a_i, F(\cdot)\rangle$ is lsc for all $i = 1, \cdots, k$. In particular, Part (a) of Corollary 3.9 applies for instance when $P = \mathbb{R}^m_+$, $F = (f_1, \cdots, f_m)$ with each $f_i : K \to \mathbb{R}$ being lsc and semistrictly quasiconvex; and (a) of Corollary 3.10 can be applied if each f_i is lsc and quasiconvex.

The following examples show that our assumptions in Corollaries 3.9 and 3.10, are in some sense optimal.

Example 3. 11 (i) The function of Example 3.7 shows the necessity of the semistrict $(\mathbb{R}^2 \setminus -\text{int } \mathbb{R}^2_+)$-quasiconvexity or semistrict $(\mathbb{R}^2 \setminus -(\mathbb{R}^2_+ \setminus \{0\}))$-quasiconvexity in (*a*) of the preceding two corollaries.

(ii) This example shows, in general, that (*a*) of Corollaries 3.9 and 3.10 fail to hold if E and E_W are unbounded. Consider $P = \mathbb{R}^2_+$, $f_1(x) = x$, $f_2(x) = -x$, $x \in \mathbb{R}$. Here $E = \mathbb{R} = E_W$ and $F = (f_1, f_2)$ is $\mathbb{R}^2_+$-convex.

(iii) In general, the reverse implication in (*a*) of Corollary 3.9 fails to hold. In fact, take $P = \mathbb{R}^2_+$, $K = [0, +\infty)$ and $f_1(x) = 1$, $f_2(x) = 2$. Here $E = K = E_W$ and $F = (f_1, f_2)$ is $\mathbb{R}^2_+$-convex.

(iv) In general, the reverse implication in (*a*) of Corollary 3.10 fails to hold. In fact, take $K = [0, +\infty)$, $P = \mathbb{R}^2_+$ and

$$f_1(x) = \begin{cases} 1, & \text{if } x \in [1,2], \\ 2 & \text{if } x \notin [1,2]. \end{cases} \quad f_2(x) = \begin{cases} 1 & \text{if } x = 4, \\ 2 & \text{if } x \neq 4. \end{cases}$$

Here $D = \{(p_1^*, p_2^*) \in \mathbb{R}^2_+ : \|(p_1^*, p_2^*)\| = 1\}$. Set $h_{p^*}(x) \doteq \langle p^*, F(x) \rangle$, $p^* \in D$, $x \in K$.
After making some computations, we obtain

$$h_{p^*}(x) = \begin{cases} 2(p_1^* + p_2^*), & \text{if } x \in [0,1[; \\ p_1^* + 2p_2^*, & \text{if } x \in [1,2[; \\ 2(p_1^* + p_2^*), & \text{if } x \in [2,4[; \\ 2p_1^* + p_2^*, & \text{if } x = 4, \\ 2(p_1^* + p_2^*), & \text{if } x \in]4, +\infty). \end{cases}$$

and

$$\text{argmin}_K h_{p^*} = \begin{cases} [1,2], & \text{if } \min\{p_1^* + 2p_2^*, 2p_1^* + p_2^*\} = p_1^* + 2p_2^*, \\ 4, & \text{if } \min\{p_1^* + 2p_2^*, 2p_1^* + p_2^*\} = 2p_1^* + p_2^*. \end{cases}$$

Here $E_W = [0, +\infty)$ and $F = (f_1, f_2)$ is $\mathbb{R}^2_+$-quasiconvex and hence semistrictly $(\mathbb{R}^2 \setminus -\text{int } \mathbb{R}^2_+)$-quasiconvex.

(v) We now show the reverse implication in (*b*) of Corollary 3.9 and (*c*) of Corollary 3.10 may be false. Take the example in (*ii*) with $K = \mathbb{R}$ to have $E = \mathbb{R}$ and $E_W = \mathbb{R}$; and consider $K = [0, +\infty)$, $f_1(x) = 1$, $f_2(x) = e^{-x}$, to have $E = \emptyset$.

Corollary 3.12 Let $\emptyset \neq K \subseteq \mathbb{R}$ be a closed convex set, $P \subseteq Y$ be a convex cone. Assume that $F : K \to Y$ is naturally *P*-quasiconvex, or equivalently, scalarly *P*-quasiconvex. Then,

$$E_W \neq \emptyset \text{ and bounded} \Rightarrow E \neq \emptyset \text{ and bounded.}$$

Proof: From Proposition 2.6 and Theorem 3.6, it follows that $R_W = \{0\}$, which implies that $R_E = \{0\}$. It turns out that E is non-empty and bounded by the same theorem.

Example 3.13 (i) The boundedness on E_W in the previous corollary cannot be deleted as the following example shows. Take $K=[0,+\infty)$, $f_1(x)=1$, $f_2(x)=e^{-x}$; one obtains $E_W=[0,+\infty]$ whereas $E=\emptyset$. Here $F=(f_1,f_2)$ is $\mathbb{R}^2_+$-convex.

(ii) The reverse implication does not hold as showed by taking $K=\mathbb{R}$. We consider $F:K\to\mathbb{R}^2$, $F=(f_1,f_2)$ with $f_1(x)=|x|$, $f_2(x)=x$. Obviously F is $\mathbb{R}^2_+$-convex (thus it is naturally $\mathbb{R}^2_+$-quasiconvex by Theorem 2.3). Here $E=\{0\}$ whereas $E_W=\mathbb{R}$.

In order to characterize the non-emptiness (possibly allowing unboundedness) of E_S, we consider the following non-coercive conditions:

(C_1) for any sequence $\{x_n\}$ in K satisfying:

(i) $|x_n|\to+\infty, \dfrac{x_n}{|x_n|}\to v\in R_S$, and,

(ii) for all $y\in K$ there exists n_y such that $F(y)-F(x_n)\in S$ for all $n\geq n_y$,

we assume the existence of $u\in K$ and $\bar{n}$, such that $|u|<|x_{\bar{n}}|$ and $F(u)-F(x_{\bar{n}})\in -S$.

(C_2) for every $x_n\in K, |x_n|\to+\infty$, there exists $\bar{n}\in\mathbb{N}$ and $u\in K$ such that $|u|<|x_{\bar{n}}|$ and $F(u)-F(x_{\bar{n}})\in -S$.

(C_3) there exists a nonempty compact set $D\subseteq K$ such that for all $x\in K\setminus D$ there exists $u\in D: F(u)-F(x)\in -S$.

(C_4) there exist $u\in K$ and $r>|u|$ such that $F(u)-F(x)\in -S$ for all $x\in K, |x|=r$.

(C_5) there exists $r>0$ such that $K_r=K\bigcap[-r,r]\neq\emptyset$ and for each $x\in K, |x|=r$ there exists $u\in K, |u|<r: F(u)-F(x)\in -S$.

We point out that all of these conditions apply to situations in which the solution set may be unbounded. Notice that the cone R_S is not explicitly mentioned in $(C_i), i=2,3,4,5$. Clearly $(C_2)\Rightarrow(C_1)$, $(C_3)\Rightarrow(C_4)$ and $(C_5)\Rightarrow(C_1)$.

We are now in a position to establish various characterizations of the nonemptiness of E_S when $K\subseteq\mathbb{R}$.

Theorem 3.14 Let $\emptyset\neq K\subseteq\mathbb{R}$ be a closed convex set, let $S\subseteq Y$ be a cone. Assume that $F:K\to Y$ is explicitly (S)-quasiconvex and Hypothesis (A) is satisfied. Then E_S is convex and each of the conditions $(C_1)-(C_5)$ is equivalent to the non-emptiness of E_S.

Proof: The convexity of E is a consequence of the remark above.

$(C_1)\Rightarrow(E_S\neq\emptyset)$: For every $n\in\mathbb{N}$, set $K_n=\{x\in K:|x|\leq n\}$. We may suppose, without loss of generality $K_n\neq\emptyset$ for all $n\in\mathbb{N}$. Let us consider the problem (13). By Hypothesis (A), it admits a solution, say $x_n\in K_n$ for all $n\in\mathbb{N}$. If $|x_n|<n$ for some $n\in\mathbb{N}$, then, we claim that x_n is also a solution to problem (10). In fact, if there is $y\in K$ with $|y|>n$ such that $F(y)-F(x_n)\notin S$. Take $z\in K$ with $z\in\text{co}(\{x_n,y\})$ and $|z|<n$. By assumption, we have $F(z)-F(x_n)\notin S$, which contradicts the choice of x_n. Therefore x_n is a solution to problem (10).

We consider now the case $|x_n|=n$ for all $n\in\mathbb{N}$. We may suppose, whithout loss of generality, that $\dfrac{x_n}{|x_n|}\to v=\pm1$. Clearly $v\in K^\infty$. We now check that $v\in R_S$. Take any $y\in K$ and $\lambda>0$. We consider the case $v=1$ (when $v=-1$ the reasoning is similar). Then $y<y+\lambda v=y+\lambda$. We choose x_n such that $x_n=n>\max\{|y+\lambda|,|y|\}$. Thus $F(y)-F(x_n)\in S$, which implies that $F(y+\lambda)-F(y)$

$\in -S$ since $y+\lambda \in (y, x_n)$. Therefore $1 = v \in R_S$. On the other hand, for any fixed $y \in K$, $F(y) - F(x_n) \in S$ for $n \in \mathbb{N}$ sufficiently large (it is enough to take $n > |y|$). This shows that x_n satisfies (i) and (ii) of condition (*). Therefore there exists $u \in K$ and $\overline{n}$, such that $|u| < |x_{\overline{n}}|$ and $F(u) - F(x_{\overline{n}}) \in -S$. We claim that $x_{\overline{n}}$ is also a solution to (10). If not, there is $y \in K$, $|y| > \overline{n} = |x_{\overline{n}}| > |u|$ such that $F(y) - F(x_{\overline{n}}) \notin S$. On the other hand, either $x_{\overline{n}} \in \mathrm{co}(\{u, y\})$ or $u \in \mathrm{co}(\{x_{\overline{n}}, y\})$ since $|u| < |x_{\overline{n}}| < |y|$. If $u \in \mathrm{co}(\{x_{\overline{n}}, y\})$, then $F(u) - F(x_{\overline{n}}) \notin S$, which contradicts the choice of $x_{\overline{n}}$. If $x_{\overline{n}} \in \mathrm{co}(\{u, y\})$ then $F(y) - F(u) \in S$, since otherwise we obtain $F(u) - F(x_{\overline{n}}) \notin -S$ contradicting the choice of u. Thus, $F(u) - F(y) \in -S$. It follows that $F(x_{\overline{n}}) - F(y) \in -S$, which contradicts the choice of y. Therefore $x_{\overline{n}}$ is a solution to problem (10).

$(C_2) \Rightarrow (C_1)$; $(C_3) \Rightarrow (C_1)$; $(C_4) \Rightarrow (C_5)$; These are obvious.

$(E_S \neq \emptyset) \Rightarrow (C_2)$: Let $|x_n| \to +\infty$ and $u \in E_S$. Thus, for $\overline{n}$ such that $|x_{\overline{n}}| > |u|$ we obtain $F(x_{\overline{n}}) - F(u) \in S$. This shows that $F(u) - F(x_{\overline{n}}) \in -S$.

$(E_S \neq \emptyset) \Rightarrow (C_3)$: Take any $u \in E_S$ and set $D = \{u\}$.

$(E_S \neq \emptyset) \Rightarrow (C_4)$: Take any $u \in E_S$ and $r > |u|$.

$(C_5) \Rightarrow (E_S \neq \emptyset)$: Let us consider problem (13) on K_r which admits a solution, say x_r. If $|x_r| < r$ we proceed as in the beginning of the proof $(C_1) \Rightarrow (E_S \neq \emptyset)$ to conclude that x_r is in fact a solution to problem (13) on K. If on the contrary $|x_r| = r$ there exists $u \in K$, $|u| < r$ such that $F(u) - F(x_r) \in -S$. We reason as in the second part of the proof as above to deduce that x_r is also solution to our original problem.

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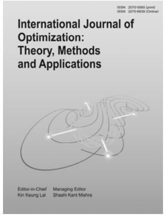

General Inexact Proximal Point Algorithm and η-Maximal Monotonicity Models with Application to Minimization

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Abstract. First, based on η-maximal monotonicity, a generalization to Rockafellar's theorem (1976) in the context of approximating a solution to a general inclusion problem involving a multivalued η-maximal monotone mapping using the inexact proximal point algorithm in a Hilbert space setting is considered. Then an application to a minimization problem of a functional on a Hilbert space is examined.The general framework for η-maximal monotonicity generalizes the general theory of multivalued maximal monotone mappings.

Keywords: inclusion problems, η-maximal monotone mapping, maximal monotone mapping, generalized resolvent operator.

1 Introduction

Let X be a real Hilbert space with the norm $\|.\|$ and the inner product $\langle\cdot,\cdot\rangle$. We consider the nonlinear inclusion problem: determine a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

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In [5], Rockafellar examined the problem of minimizing a lower semicontinuous proper convex functional *f* on a Hilbert space setting, where the proximal point algorithm

$$x^{k+1} \approx P_k(x^k), \tag{2}$$

in exact form generates a sequence $\{x^k\}$ by taking x^{k+1} to be the minimizer of

$$f(x)+\frac{1}{\rho_k}\| x-x^k \|^2, \tag{3}$$

where $\rho_k > 0$.

Indeed, the inclusion problem (1) encompasses a general class of problems of variational nature, such as minimization or maximization of functions, variational inequality problems, and minimax problems. The maximal monotonicity/ generalized maximal monotonicity has been a powerful tool to studying convex programming and variational inequalities. It also turned out that one of the fundamental algorithms applied for solving these problems was in fact the proximal point algorithm and its variant forms. Furthermore,, Rockafellar [6] applied the proximal point algorithm in convex programming. For more details, we refer the reader to [1-22].

In this paper, we first intend to examine some results involving η-maximal monotone and generalized firmly nonexpansive mappings. Second, we generalize Rockafellar's theorem [5] to the case η-maximal monotone mappings. Finally, we apply this generalization to a minimization problem. The obtained results, in turn, generalize a general class of results, including the investigations involving maximal monotone mappings.

2 η-Maximal Monotonicity Framework

This section deals with some results based on basic properties of η-maximal monotonicity and other results involving η-maximal monotonicity and the generalized firm nonexpansiveness. Let *X* denote a real Hilbert space with the norm $\|.\|$ and inner product $<\cdot,\cdot>$. Let $M : X \to 2^X$ be a multivalued mapping on *X*. We shall denote both the map M and its graph by *M*, that is, the set $\{(x,y): y \in M(x)\}$. This is equivalent to stating that a mapping is any subset *M* of $X \times X$, and $M(x) = \{y:(x,y) \in M\}$. If *M* is single-valued, we shall still use $M(x)$ to represent the unique *y* such that $(x,y) \in M$ rather than the singleton set $\{y\}$. This interpretation shall much depend on the context. The domain of a map *M* is defined (as its projection onto the first argument) by

$$D(M) = \{x \in X : \exists y \in X : (x,y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

D(M)=X, shall denote the full domain of *M*, and the range of *M* is defined by

$$R(M) = \{y \in X : \exists x \in X : (x,y) \in M\}.$$

The inverse M^{-1} of *M* is $\{(y,x):(x,y) \in M\}$. For a real number ρ and a mapping *M*, let $\rho M = \{x, \rho y):(x,y) \in M\}$. If *L* and *M* are any mappings, we define

$$L+M=\{(x,y+z):(x,y)\in L,(x,z)\in M\}.$$

Definition 2.1. A mapping $\eta: X\times X\to X$ is said to be:

(i) monotone if

$$\langle x-y,\eta(x,y)\rangle\geq 0\,\forall\, x,y\in X.$$

(ii) (r)-strongly monotone if there exists a positive constant r such that

$$\langle x-y,\eta(x,y)\rangle\geq r\,\|x-y\|^2\,\forall\, x,y\in X.$$

(iii) τ-Lipschitz continuous if there is a positive constant τ such that

$$\|\eta(x,y)\|\leq\tau\,\|x-y\|\,\forall\, x,y\in X.$$

Definition 2.2. Let $\eta: X\times X\to X$ be a mapping, and let $M: X\to 2^X$ be a multivalued mapping on *X*. The map *M* is said to be:

(i) η-monotone if

$$\|u^*-v^*,\eta(u,v)\|\geq 0\,\forall\,(u,u^*),(v,v^*)\in M.$$

(ii) η-pseudomonotone if

$$\langle v^*,\eta(u,v)\rangle\geq 0\Rightarrow\langle u^*,\eta(u,v)\rangle\geq 0\,\forall\,(u,u^*),(v,v^*)\in M.$$

(iii) (r,η)-strongly monotone if there exists a positive constant *r* uch that

$$\langle u^*-v^*,\eta(u,v)\rangle\geq r\,\|u-v\|^2\,\forall\,(u,u^*),(v,v^*)\in M.$$

(iv) (m,η)-relaxed monotone if there exists a positive constant m such that

$$\langle u^*-v^*,\eta(u,v)\rangle\geq(-m)\,\|u-v\|^2\,\forall\,(u,u^*),(v,v^*)\in M.$$

Definition 2.3. Let $\eta: X\times X\to X$ be a mapping, and let $M: X\to 2^X$ be a multivalued mapping on *X*. e map *M* s said to be:

(i) Nonexpansive if

$$\|u^*-v^*\|\leq\|u-v\|\,\forall\,(u,u^*),(v,v^*)\in M.$$

(ii) η-firmly nonexpansive if

$$\|u^*-v^*\|^2\leq\langle u^*-v^*,\eta(u,v)\rangle\,\forall\,(u,u^*),(v,v^*)\in M.$$

(iii) (c,η)-irmly nonexpansive if there exists a constant $c>0$ such that

$$\|u^*-v^*\|^2\leq c\langle u^*-v^*,\eta(u,v)\rangle\,\forall\,(u,u^*),(v,v^*)\in M.$$

Definition 2.4. The map $M : X \to 2^X$ is said to be η-maximal monotone if

1. M is η-monotone
2. $R(I + \rho M) = X$ for $\rho > 0$,

where I is the identity mapping.

Definition 2.5. Let $M : X \to 2^X$ be an η-maximal monotone mapping. Then the generalized resolvent operator $J^M_{\rho,\eta} : X \to X$ is defined by

$$J^M_{\rho,\eta}(u) = (I + \rho M)^{-1}(u).$$

Proposition 2.1. Let $M : X \to 2^X$ be an η--maximal monotone mapping. Then the operator $(I + \rho M)^{-1}$ is single-valued.

3 η-Inclusion Problems

This section deals with a generalization to Rockafellar's theorem [5, Theorem 1] under the framework of η-maximal monotonicity. Furthermore, some results connecting η- maximal monotonicity and corresponding generalized resolvent operator are established.

Lemma 3.1. ([14]) Let X be a real Hilbert space, and let $M : X \to 2^X$ be η-maximal monotone. Furthermore, let $\eta : X \times X \to X$ be (t)-strongly monotone and τ-Lipschitz continuous. Then the generalized resolvent operator associated with M and defined by

$$J^M_{\rho,\eta}(u) = (I + \rho M)^{-1}(u) \,\forall u \in X,$$

is $(\frac{\tau}{t})$ -Lipschitz continuous, that is,

$$\| J^M_{\rho,\eta}(u) - J^M_{\rho,\eta}(v) \| \le \frac{\tau}{t} \| u - v \|.$$

Lemma 3.2. Let X be a real Hilbert space, and let $M : X \to 2^X$ be η-maximal monotone. Let $\eta : X \times X \to X$ be (t)-strongly monotone.Then the generalized resolvent operator associated with M and defined by

$$J^M_{\rho,\eta}(u) = (I + \rho M)^{-1}(u) \,\forall u \in X,$$

satisfies

$$\| J^M_{\rho,\eta}(u) - J^M_{\rho,\eta}(v) \|^2 \le \frac{1}{t} \left\langle u - v, \eta(J^M_{\rho,\eta}(u), J^M_{\rho,\eta}(v)) \right\rangle.$$

Proof. For any $u, v \in X$, using the definition of the resolvent operator $J^M_{\rho,\eta}$, we have

$$\frac{1}{\rho}[u - J^M_{\rho,\eta}(u)] \in M(J^M_{\rho,\eta})(u),$$

and

$$\frac{1}{\rho}[v - J^M_{\rho,\eta}(v)] \in M(J^M_{\rho,\eta})(v).$$

Given that M is η-monotone, we have

$$\frac{1}{\rho}\left\langle u - v - (J^M_{\rho,\eta}(u) - J^M_{\rho,\eta}(v)), \eta(J^M_{\rho,\eta}(u), J^M_{\rho,\eta}(v))\right\rangle \geq 0. \tag{4}$$

It follows from (4) that

$$\begin{aligned}&\left\langle u - v, \eta(J^M_{\rho,\eta}(u), J^M_{\rho,\eta}(v))\right\rangle \\ &\geq \left\langle J^M_{\rho,\eta}(u) - J^M_{\rho,\eta}(v), \eta(J^M_{\rho,\eta}(u), J^M_{\rho,\eta}(v))\right\rangle \\ &\geq t \| J^M_{\rho,\eta}(u) - J^M_{\rho,\eta})(v) \|^2 .\end{aligned}$$

Theorem 3.1. Let X be a real Hilbert space, and let $M : X \to 2^X$ be η-maximal monotone. Then the following statements are mutually equivalent:

(i) An element $u \in X$ is a solution to (1).
(ii) For an $u \in X$, we have

$$u = J^M_{\rho,\eta}(u),$$

where

$$J^M_{\rho,\eta}(u) = (I + \rho M)^{-1}(u).$$

Proof. It follows from the definition of η-resolvent operator corresponding to M.

Proposition 3.1. Let X be a real Hilbert space, and let $M : X \to 2^X$ be η-maximal monotone. Let $\eta : X \times X \to X$ be $(t)-$strongly monotone and $(\tau)-$Lipschitz continuous. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the proximal point algorithm

$$x^{k+1} \approx J^M_{\rho_k,\eta}(x^k) \tag{5}$$

such that

$$\| x^{k+1} - J^M_{\rho_k,\eta}(x^k) \| \leq \varepsilon_k, \tag{6}$$

for $\Sigma_{k=0}^{\infty}\varepsilon_k < \infty$, and $\{\rho_k\}$ is bounded away from zero, and for $\{\varepsilon_k\}, \{\rho_k\} \subseteq [0, \infty)$,

$$dist(0, S_k^*(x^{k+1})) \leq \frac{\varepsilon_k}{\rho_k}, \tag{7}$$

where $S_k^*(x) = M(x) + \rho_k^{-1}(x - x^k)$ and $\Sigma_{k=0}^{\infty}\varepsilon_k < \infty$,
and the estimate

$$\| x^{k+1} - J^M_{\rho_k,\eta}(x^k) \| \leq \frac{\rho_k \tau}{t} dist(0, S_k^*(x^{k+1})), \quad (8)$$

with $J^M_{\rho_k,\eta} = (I + \rho_k M)^{-1}$, and $\{\varepsilon_k\}, \{\rho_k\} \subseteq [0, \infty)$. Then (7) implies (8), and the estimate holds.

Proof. To show, (7) implies (8), let $w \in S_k^*(x^{k+1})$. Then

$$\rho_k w + x^k \in (I + \rho_k M)(x^{k+1}),$$

so we have

$$x^{k+1} = (I + \rho_k M)^{-1}(\rho_k w + x^k) = J^M_{\rho,\eta}(\rho_k w + x^k).$$

Applying Lemma 3.1, we get

$$\begin{aligned} &\| x^{k+1} - J^M_{\rho_k,\eta}(x^k) \| \\ &= \| J^M_{\rho_k,\eta}(\rho_k w + x^k) - J^M_{\rho_k,\eta}(x^k) \| \\ &\leq \frac{\rho_k \tau}{t} \| w \|. \end{aligned}$$

It follows that

$$\| x^{k+1} - J^M_{\rho_k,\eta}(x^k) \| \leq \frac{\rho_k \tau}{t} min\{\| w \| : w \in S_k^*(x^{k+1})\},$$

which completes the proof.

We observe that Proposition 3.1, unlike in the case of Rockafellar [5], is not suitable to achieve the weak convergence of the sequence.

Proposition 3.2. Let X be a real Hilbert space, and let $M : X \to 2^X$ be η-maximal monotone. Let $\eta : X \times X \to X$ be (t)-strongly monotone. If, in addition,

$$\left\langle u - v, J^M_{\rho,\eta}(u) - J^M_{\rho,\eta}(v) \right\rangle \geq \gamma \left\langle u - v, \eta(J^M_{\rho,\eta}(u), J^M_{\rho,\eta}(v)) \right\rangle \forall u, v \in X,$$

then, for $J_k^* = I - J^M_{\rho,\eta}$, we have (for $\gamma t > 1$)

$$\left\langle u - v, J_k^*(u) - J_k^*(v) \right\rangle \geq \frac{\gamma t - 1}{2\gamma t - 1} \| u - v \|^2 + \frac{\gamma t}{2\gamma t - 1} \| J_k^*(u) - J_k^*(v) \|^2,$$

where $J^M_{\rho_k,\eta} = (I + \rho M)^{-1}$.

Proof. The proof follows from using Lemma 3.2, and a simple manipulation.

Theorem 3.2. Let X be a real Hilbert space, and let $M : X \to 2^X$ be η-maximal monotone. Let $\eta : X \times X \to X$ be (t)-strongly monotone and weakly continuous in the second argument. Suppose that (for $\gamma > 0$)

$$\left\langle u-v, J^M_{\rho,\eta}(u)-J^M_{\rho,\eta}(v)\right\rangle \geq \gamma\left\langle u-v, \eta(J^M_{\rho,\eta}(u), J^M_{\rho,\eta}(v))\right\rangle \forall u,v\in X. \tag{9}$$

For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the proximal point algorithm

$$x^{k+1} \approx J^M_{\rho_k,\eta}(x^k) \tag{10}$$

such that

$$\| x^{k+1} - J^M_{\rho_k,\eta}(x^k)\| \leq \varepsilon_k, \tag{11}$$

where $J^M_{\rho_k,\eta} = (I+\rho_k M)^{-1}$, and

$$\{\varepsilon_k\}, \{\rho_k\} \subseteq [0,\infty)$$

are scalar sequences with $e_1 = \Sigma^{\infty}_{k=0}\varepsilon_k < \infty$, and $\{\rho_k\}$ is bounded away from zero.

Furthermore suppose that the sequence $\{x^k\}$ is bounded in the sense that there exists at least one solution to $0 \in M(x)$. Then the following conclusions hold:

(i) For $\gamma t > 1$, we have

$$(2\gamma t - 1)\| J^M_{\rho,\eta}(x^k) - x^* \|^2 \leq \| x^k - x^*\|^2 - \| J^*_k(x^k)\|^2 .$$

(ii) The sequence $\{x^k\}$ is bounded.

(iii) $lim_{k\to\infty} J^*_k(x^k) = lim_{k\to\infty} \| x^{k+1} - x^k \| = 0$ for $\rho_k > 0$.

(iv) The sequence $\{x^k\}$ converges weakly to a solution of (1).

Proof. Suppose that x^* is a zero of M. For all $k \geq 0$, we set

$$J^*_k = I - J^M_{\rho_k,\eta}.$$

Then from Theorem 3.1, it follows that any solution to (1) is a fixed point of $J^M_{\rho_k,\eta}$, and hence a zero of J^*_k. The proof of (i) follows from Lemma 3.2 and the following equality

$$\| u-v\|^2 = \| J^M_{\rho,\eta}(u) - J^M_{\rho,\eta})(v) + J^*_k(u) - J^*_k)v)\|^2 \ \forall u,v\in X.$$

Next, we find the estimate

$$\begin{aligned} &\| x^{k+1} - x^*\| - \varepsilon_k \leq \| x^{k+1} - x^* \| - \| x^{k+1} - J^M_{\rho_k,\eta}(x^k)\| \\ &\leq \| x^{k+1} - x^* - (x^{k+1} - J^M_{\rho_k,\eta}(x^k))\| \\ &= \| J^M_{\rho_k,\eta}(x^k) - x^*\| \\ &= \| x^k - x^* - (x^k - J^M_{\rho_k,\eta}(x^k)\| \\ &= \| x^k - x^* - J^*_k(x^k)\|. \end{aligned} \tag{12}$$

Applying Proposition 3.2, for $\gamma t > 1$, we have

$$\begin{aligned}
&\| x^k - x^* - J_k^*(x^k) \|^2 \\
&= \|x^k - x^*\|^2 - 2\left\langle x^k - x^*, J_k^*(x^k) \right\rangle + \|J_k^*(x^k)\|^2 \\
&\leq (1 - \frac{2(\gamma t - 1)}{2\gamma t - 1})\|x^k - x^*\|^2 - (\frac{2\gamma t}{2\gamma t - 1} - 1)\|J_k^*(x^k)\|^2 \\
&\leq \|x^k - x^*\|^2 - (\frac{2\gamma t}{2\gamma t - 1} - 1)\|J_k^*(x^k)\|^2.
\end{aligned}$$

Since $\left(\frac{2\gamma t}{2\gamma t - 1} - 1\right) > 0,$ and $0 < \frac{2(\gamma t - 1)}{2\gamma t - 1} < 1,$ we have

$$\| x^{k+1} - x^* \| - \varepsilon_k \leq \| x^k - x^* \|.$$

It further follows that

$$\| x^{k+1} - x^* \| \leq \| x^k - x^* \| + \varepsilon_k. \tag{13}$$

Combining (13) for all k, we have

$$\begin{aligned}
&\| x^{k+1} - x^* \| \leq \| x^0 - x^* \| + \sum_{i=0}^{k} \varepsilon_i \\
&\leq \| x^0 - x^* \| + e_1.
\end{aligned} \tag{14}$$

Hence, the sequence $\{x^k\}$ is bounded.
Furthermore, it follows from (14) that

$$\begin{aligned}
&\| x^k - x^* \| \leq \| x^0 - x^* \| + \sum_{i=1}^{k} \varepsilon_{i-1} \\
&\leq \| x^0 - x^* \| + e_1.
\end{aligned} \tag{15}$$

Next, we estimate

$$\begin{aligned}
&\| x^{k+1} - x^* \|^2 \leq \| x^k - x^* - J_k^*(x^k) \|^2 + \varepsilon_k^2 \\
&+ 2\|x^k - x^* - J_k^*(x^k)\|\varepsilon_k \\
&\leq \|x^k - x^*\|^2 - (\frac{2\gamma t}{2\gamma t - 1} - 1)\|J_k^*(x^k)\|^2 + \varepsilon_k^2 \\
&+ 2\varepsilon_k\|x^k - x^*\| \\
&\leq \| x^k - x^* \|^2 - (\frac{2\gamma t}{2\gamma t - 1} - 1)\| J_k^*(x^k) \|^2 + \varepsilon_k^2 \\
&+ 2\varepsilon_k(\|x^0 - x^*\| + e_1) \\
&\leq \| x^0 - x^* \|^2 - (\frac{2\gamma t}{2\gamma t - 1} - 1)\sum_{i=0}^{k} \| J_i^*(x^i) \|^2 + e_2 \\
&+ 2e_1(\|x^0 - x^*\| + e_1),
\end{aligned}$$

where $e_2 = \sum_{k=0}^{\infty} \varepsilon_k^2 < \infty$ since $\{\varepsilon_k\}$ is summable (and hence $\{\varepsilon_k^2\}$ is summable as well).
As $k \to \infty$, we infer that

$$\sum_{i=0}^{k} \| J_i^*(x^i) \|^2 < \infty \text{ implies } lim_{k\to\infty} J_k^*(x^k) = 0.$$

This implies that

$$\rho_k^{-1} \| J_k^*(x^k) \| \to 0,$$

where ρ_k is bounded away from zero. Since the sequence $\{x^k\}$ is bounded, it must have at least one weak cluster point, say x'. It follows that there exists a unique element $(u^k, v^k) \in M$ represented by $u^k + \rho_k v^k = x^k$ for all k. Since $u^k = J_{\rho_k,\eta}^M(x^k)$ and $lim_{k\to\infty} J_k^*(x^k) = 0$, it implies that $x^k - u^k \to 0$ as $k \to \infty$. It further follows that

$$lim_{k\to\infty} v^k = lim_{k\to\infty} \frac{J_k^*(x^k)}{\rho_k} = 0.$$

Let $\{x^{k(j)}\}$ be a subsequence of $\{x^k\}$ such that $x^{k(j)}$ converges weakly to x'. Since $x^k - u^k \to 0$, it implies $u^{k(j)}$ also converges weakly to x'. Let some $(u,v) \in M$. Then the η-maximal monotonicity of M implies that

$$\left\langle \eta(u,u^k), v - v^k \right\rangle \geq 0 \; for\; all\; k \geq 0.$$

Therefore, using the weak continuity of η in second argument, we have

$$\left\langle \eta(u,x'), v - 0 \right\rangle \geq 0 \; for\; all\; k \geq 0.$$

Since M is η-monotone, and (u,v) is arbitrary, it follows that $(x',0) \in M$, that is, x' is a solution to (1).

Finally, it turns out that the weak cluster point of the sequence $\{x^k\}$ is unique under the assumptions of the theorem.

4 Application to a Minimization Problem

In this section, we examine a minimization problem to the context of the η-maximal monotonicity.

Definition 4.1. Let $f : X \to (-\infty, +\infty]$ be a lower semicontinuous proper convex functional on X. The functional $u^* \in X$ is said to be a *subgradient* of *f* at a point u iff

(i) $f(u) \not\equiv +\infty$,
(ii) inequality

$$f(v) - f(u) \geq \left\langle u^*, \eta(v,u) \right\rangle \forall v \in X. \tag{16}$$

The set of all subgradients of *f* at *u* is called the *subdifferential*, $\partial f(u)$ of *f* at *u*. Let $f : X \to (-\infty, +\infty]$ be a locally Lipschitz functional on X, and let $\eta : X \times X \to X$ be (t)strongly monotone and (τ)-Lipschitz continuous such that ∂f is η-monotone, that is,

$$\langle u^* - v^*, \eta(u,v)\rangle \geq 0,$$

where $u^* \in \partial f(u)$ and $v^* \in \partial f(v)$. Then ∂f is $\eta-$ maximal monotone such that (for $\forall z^*$)

$$w \in \partial f(x) \Leftrightarrow f(x^*) - f(x) \geq \langle \eta(x^*, x), w\rangle$$
$$\Leftrightarrow x \in arg\ min(f - \langle ., w\rangle).$$

This, in special case, is equivalent to stating that

$$0 \in \partial f(x) \Leftrightarrow x \in arg\ min\, f.$$

Thus, the generalized proximal point algorithm for $M = \partial f$ provides a minimizing method for *f*.

Now we are preparing for an application of Theorem 3.2 to the minimizing problem. A function $\phi : X \to (-\infty, +\infty]$ is said to be (α)-strongly convex if there exists a positive constant α such that

$$\phi((1-\lambda)x + \lambda x^*) \leq (1-\lambda)\phi(x) + \lambda\phi(x^*) - \frac{1}{2}\alpha\lambda(1-\lambda)\| x - x^* \|^2,$$

where $x, x^* \in X$ and $0 < \lambda < 1$.

For an arbitrarily chosen initial point x^0, the the proximal point algorithm

$$x^{k+1} \approx J^M_{\rho_k,\eta}(x^k) \tag{17}$$

generates the sequence $\{x^k\}$ where $J^M_{\rho_k,\eta} = (I + \rho_k M)^{-1}$, and $\{\rho_k\}$ is a sequence of positive real numbers. When $M = \partial f$, this iterative procedure reduces to

$$x^{k+1} \approx arg\, min_x \phi_k(x), \tag{18}$$

where

$$\phi_k(x) = f(x) + \frac{t}{2\rho_k \tau}\| x - x^k \|^2\ . \tag{19}$$

Theorem 4.1. Let $M = \partial f$, and let the map ϕ_k be lower semicontinuous and $(\frac{t}{\rho_k \tau})$-strongly convex. Furthermore, suppose that the sequence $\{x^k\}$ is generated by the proximal algorithm

$$x^{k+1} \approx J^M_{\rho_k,\eta}(x^k)$$

under the assumption of Theorem 3.1, which satisfies the condition (7). Then we arrive at the following conclusions:

(i) $S^*_k = \partial \phi_k$ in condition (7).

(ii) $\partial \phi_k$ is lower semicontinuous and $(\frac{t}{\rho_k \tau})$-strongly convex, where

(iii) $x^k \to x^*$ weakly, where $f(x^*)$ = min f.

(iv) *f* satisfies

$$f(x^{k+1}) - f(x^*) \leq t(\rho_k \tau)^{-1} \| x^{k+1} - x^* \| (\varepsilon_k + \| x^{k+1} - x^k \|) \to 0.$$

Proof. To prove (i) if we just subdifferentiate ϕ_k both sides of (19), we get

$$\partial\phi_k(x) = \partial f(x) + \frac{t}{\rho_k \tau} \| x - x^k \| \equiv S_k^*(x) \ \forall x.$$

For the proof of (ii), the definition of ϕ_k ensures the lower semicontinuity and $(\frac{t}{\rho_k \tau})$ -strong convexity.

Next we prove (iii)-(iv) as follows: Since, in light of condition (19), a closed convex set $\partial\phi_k(x^{k+1})$ is nonempty, there exists a unique element $w^k \in \partial\phi_k(x^{k+1})$ such that

$$w^k - \frac{t}{\rho_k \tau}(x^{k+1} - x^k) \in \partial f(x^{k+1}),$$

where

$$\| w^k \| \leq \frac{\varepsilon_k t}{\rho_k \tau} \to 0. \tag{20}$$

Based on Theorem 3.2, let x^* be the weak limit of the sequence $\{x^k\}$. Then, for $0 \in \partial f(x^*)$ and using the subgradient inequality, we have

$$f(x^{k+1}) + \left\langle w^k - (\frac{t}{\rho_k \tau})(x^{k+1} - x^k), \eta(x^*, x^{k+1}) \right\rangle \leq f(x^*) = min\, f.$$

It further follows that

$$f(x^{k+1}) - f(x^*) \leq \left\langle w^k - (\frac{t}{\rho_k \tau})(x^{k+1} - x^k), \eta(x^{k+1}, x^*) \right\rangle$$

$$\leq \| \eta(x^{k+1}, x^*) \| \| w^k - (\frac{t}{\rho_k \tau})(x^{k+1} - x^k) \|$$

$$\leq \| \eta(x^{k+1}, x^*) \| [\| w^k \| + (\frac{t}{\rho_k \tau}) \| x^{k+1} - x^k \|]$$

$$\leq \tau \| x^{k+1} - x^* \| [\| w^k \| + (\frac{t}{\rho_k \tau}) \| x^{k+1} - x^k \|]$$

$$\leq \frac{t}{\rho_k} \| x^{k+1} - x^* \| [\varepsilon_k + \| x^{k+1} - x^k \|]$$

Finally, on applying Theorem 3.2 and (20), we have

$$f(x^{k+1}) - f(x^*) \leq \frac{t}{\rho_k} \| x^{k+1} - x^* \| [\varepsilon_k + \| x^{k+1} - x^k \|] \to 0.$$

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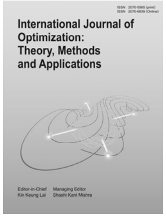

On the Sum of Two Triangular Random Variables

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Abstract. In this note, we derive the probability density function (pdf) for the sum of two independent triangular random variables having different supports, by considering all possible cases. As an illustration, we obtain the value of pdf, for some selected values of the parameters, involved in their definitions. We also give some graphs showing variation in pdf with variations in the mode and two boundary points.

Keywords and phrases: triangular distribution, Laplace transforms, distribution of sum of independent random variables.

1 Introduction

The triangular distribution is often used when there is only limited sample data, and especially in cases where the relationship between variables is known but data is scarce. In 1997, Johnson explored the advantages of using the triangular distribution as a proxy for the beta distribution. Advantages of using triangular distribution over beta distribution have been discussed in detail in the book by Kotz and Van Dorp (2004). Recently the triangular distribution is gaining popularity due to its use in discrete system simulation and Monte Carlo simulation technique as noted by Banks *et al.*(2000) and Wright (2002) respectively. Antonia and Goncalves (2006) add some new applications of triangular and trapezoidal distributions in the genome analysis, particularly, in the construction of physical

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mapping of linear and circular chromosomes. The distribution of sum of two triangular random variables X and Y is used in PERT analysis as referred by Johnson (1997) where the random variables X and Y may be viewed as uncertainty of two independent consecutive activity durations and X+Y is the overall completion time.

In this note, we derive the pdf of the sum of two non identical, independent and triangularly distributed random variables. The distribution of sum of random variables is of interest in many areas of physics and engineering. For example, sums of independent gamma random variables have application in problems of queuing theory such as determination of total waiting time, in civil engineering, such as determination of the total excess water flow in a dam. Since 1934, the problem of deriving the distribution of sums of random variables has received a great deal of attention and systematic procedures for determining such distributions have been well developed by many authors such as Witner (1934), Aroian (1944), Cramer (1946, 1962) and Lukacs (1964, 1970) to mention a few. The distribution of sum of two independent random variables has been obtained by several authors. In this context the works of Albert (2002) for uniform variates, Moschopoulos (1985) and Holm (2004) for gamma variates, Loaiciga and Leipnik (1999) for Gumble variates, Van Dorp and Kotz (2003) for triangular variates (sharing the same support but not necessarily the same mode) are worth mentioning. Recently Garg (2008) has obtained the distribution of sum of triangular and bilateral exponential random variables. Here we aim to extend the work of Dorp and Kotz (2003) by taking two triangular random variables lying on different supports.

2 Derivation of Pdf

A triangular distribution of a random variable X_1 is a continuous probability distribution with lower limit a_1, mode m_1 and upper limit b_1. Its pdf on the support $a_1 \le x_1 \le b_1$ is defined as

$$f_1(x_1) = \begin{cases} \dfrac{2(x_1 - a_1)}{(b_1 - a_1)(m_1 - a_1)} & a_1 \le x \le m_1 \\ \dfrac{2(b_1 - x_1)}{(b_1 - a_1)(b_1 - m_1)} & m_1 \le x \le b_1 \end{cases} \tag{1}$$

We take one more independent and triangularly distributed random variable X_2 on the support $a_2 \le x_2 \le b_2$ having mode m_2. To simplify the presentation, we assume without loss of generality that both X_1 and X_2 are positive.

The pdf of the sum of two random variables $W = X_1+X_2$ can be obtained by using Laplace transform and its inversion noted by Springer (1979) and is expressed as

$$h(w) = L_s^{-1}\left[L_s f_1(x_1) L_s f_2(x_2)\right] \tag{2}$$

where $f_1(x_1)$ and $f_2(x_2)$ denote the pdf's of random variables X_1 and X_2 respectively. The Laplace transform of $f_1(x_1)$ can easily be obtained as

$$L_s(f_1(x_1)) = \begin{cases} \dfrac{2}{(b_1-a_1)(m_1-a_1)}\left((m_1-a_1)\dfrac{e^{-sm_1}}{-s} - \dfrac{e^{-sm_1}}{s^2} + \dfrac{e^{-sa_1}}{s^2}\right) & a_1 \le x_1 \le m_1 \\ \dfrac{2}{(b_1-a_1)(b_1-m_1)}\left((b_1-m_1)\dfrac{e^{-sm_1}}{s} - \dfrac{e^{-sm_1}}{s^2} + \dfrac{e^{-sb_1}}{s^2}\right) & m_1 \le x_1 \le b_1 \end{cases} \tag{3}$$

Similarly

$$L_s(f_2(x_2)) = \begin{cases} \dfrac{2}{(b_2-a_2)(m_2-a_2)}\left((m_2-a_2)\dfrac{e^{-sm_2}}{-s} - \dfrac{e^{-sm_2}}{s^2} + \dfrac{e^{-sa_2}}{s^2}\right) & a_2 \le x_2 \le m_2 \\ \dfrac{2}{(b_2-a_2)\,(b_2-m_2)}\left((b_2-m_2)\dfrac{e^{-sm_2}}{s} - \dfrac{e^{-sm_2}}{s^2} + \dfrac{e^{-sb_2}}{s^2}\right) & m_2 \le x_2 \le b_2 \end{cases} \tag{4}$$

Now, derivation of the pdf $h(w)$ requires consideration of four different cases where the values of X_1 and X_2 are located in different segments of their domains. The four cases along with their sub cases are shown in the Table 1.

Table 1. Cases and sub cases involved in determining the pdf of sum

S. No.	Cases	Sub-Cases
I	$a_1 \le x_1 \le m_1$ and $a_2 \le x_2 \le m_2$	$a_1+m_2 < m_1+a_2$, $a_1+m_2 = m_1+a_2$, $a_1+m_2 > m_1+a_2$
II	$a_1 \le x_1 \le m_1$ and $m_2 \le x_2 \le b_2$	$a_1+b_2 < m_1+m_2$, $a_1+b_2 = m_1+m_2$, $a_1+b_2 > m_1+m_2$
III	$m_1 \le x_1 \le b_1$ and $a_2 \le x_2 \le m_2$	$m_1+m_2 < b_1+a_2$, $m_1+m_2 = b_1+a_2$, $m_1+m_2 > b_1+a_2$
IV	$m_1 \le x_1 \le b_1$ and $m_2 \le x_2 \le b_2$	$m_1+b_2 < b_1+m_2$, $m_1+b_2 = b_1+m_2$, $m_1+b_2 > b_1+m_2$

Results

Case I. $a_1 \le x_1 \le m_1$ and $a_2 \le x_2 \le m_2$

We now obtain the expression for $h_1(w)$, the value of $h(w)$ in Case I. Using eq. (3) and (4) in eq. (2), we get

$$h_1(w) = L_s^{-1}\left[k_1\left((m_1-a_1)\frac{e^{-sm_1}}{-s} - \frac{e^{-sm_1}}{s^2} + \frac{e^{-sa_1}}{s^2}\right)\left((m_2-a_2)\frac{e^{-sm_2}}{-s} - \frac{e^{-sm_2}}{s^2} + \frac{e^{-sa_2}}{s^2}\right)\right] \tag{5}$$

where $k_1 = \dfrac{4}{(b_1-a_1)(b_2-a_2)(m_1-a_1)(m_2-a_2)}$

On taking the inverse Laplace transform of the functions in the r.h.s. of (5), the value of $h_1(w)$, for different values of can be written as follows.

$$h_1(w)=k_1\left[\frac{w^3}{6}-\frac{w^2}{2}(a_1+a_2)-w\left(\frac{m_1^2}{2}+\frac{m_2^2}{2}-a_1m_1-a_2m_2-a_1a_2\right)-\frac{m_1^2}{2}(a_1-a_2)\right.$$
$$\left.+\frac{m_2^2}{2}(a_1-a_2)-a_1a_2(m_1+m_2)+\frac{m_1^3}{3}+\frac{m_2^3}{3}\right] \tag{6}$$

for $w\geq m_1+m_2$

$$h_1(w)=k_1\left[-\frac{w^3}{6}+\frac{w^2}{2}(a_1+a_2)+w\left(-\frac{a_1^2}{2}+\frac{m_2^2}{2}-a_2m_2-a_1a_2\right)\right.$$
$$\left.+\frac{m_2^2}{2}(a_2-a_1)+\frac{a_1^2a_2}{2}+a_1a_2m_2+\frac{a_1^3}{6}-\frac{m_2^3}{3}\right] \tag{7}$$

for $w\geq a_1+m_2$

$$h_1(w)=k_1\left[-\frac{w^3}{6}+\frac{w^2}{2}(a_1+a_2)+w\left(\frac{m_1^2}{2}-\frac{a_2^2}{2}-a_1m_1-a_1a_2\right)\right.$$
$$\left.+\frac{m_1^2}{2}(a_1-a_2)+\frac{a_1a_2^2}{2}+a_1a_2m_1+\frac{a_2^3}{6}-\frac{m_1^3}{3}\right] \tag{8}$$

for $w\geq m_1+a_2$

$$h_1(w)=k_1\left[\frac{w^3}{6}-\frac{w^2}{2}(a_1+a_2)+\frac{w}{2}(a_1+a_2)^2-\frac{1}{6}(a_1+a_2)^3\right]\quad \text{for}\quad w\geq a_1+a_2 \tag{9}$$

Combining these equations according to the sub cases of Case I, the following expressions for $h_1(w)$ are obtained on the support $[a_1+a_2, m_1+m_2]$.

If $a_1+m_2<m_1+a_2$:

$$h_1(w)=\begin{cases} k_1\times\left[\frac{w^3}{6}-\frac{w^2}{2}(a_1+a_2)+\frac{w}{2}(a_1+a_2)^2-\frac{1}{6}(a_1+a_2)^3\right] & w\in(a_1+a_2,a_1+m_2) \\ k_1\times\left[\frac{w}{2}(m_2-a_2)^2-\frac{m_2^2}{2}(a_1-a_2)-\frac{a_1a_2^2}{2}+a_1a_2m_2-\frac{a_2^3}{6}-\frac{m_2^3}{3}\right] & w\in(a_1+m_2,m_1+a_2) \\ k_1\times\left[-\frac{w^3}{6}+\frac{w^2}{2}(a_1+a_2)+w\left(\frac{m_1^2}{2}+\frac{m_2^2}{2}-a_1m_1-a_2m_2-a_1a_2\right)\right. & \\ \left.+\frac{m_1^2}{2}(a_1-a_2)-\frac{m_2^2}{2}(a_1-a_2)+a_1a_2(m_1+m_2)-\frac{m_1^3}{3}-\frac{m_2^3}{3}\right] & w\in(m_1+a_2,m_1+m_2) \end{cases} \tag{10}$$

If $a_1+m_2=m_1+a_2$:

$$h_1(w)=\begin{cases} k_1\times\left[\frac{w^3}{6}-\frac{w^2}{2}(a_1+a_2)+\frac{w}{2}(a_1+a_2)^2-\frac{1}{6}(a_1+a_2)^3\right] & w\in(a_1+a_2,a_1+m_2) \\ k_1\times\left[-\frac{w^3}{6}+\frac{w^2}{2}(a_1+a_2)+w\left(\frac{m_1^2}{2}+\frac{m_2^2}{2}-a_1m_1-a_2m_2-a_1a_2\right)\right. & \\ \left.+\frac{m_1^2}{2}(a_1-a_2)-\frac{m_2^2}{2}(a_1-a_2)+a_1a_2(m_1+m_2)-\frac{m_1^3}{3}-\frac{m_2^3}{3}\right] & w\in(a_1+m_2,m_1+m_2) \end{cases} \tag{11}$$

If $a_1+m_2>m_1+a_2$:

$$h_1(w)=\begin{cases} k_1\times\left[\frac{w^3}{6}-\frac{w^2}{2}(a_1+a_2)+\frac{w}{2}(a_1+a_2)^2-\frac{1}{6}(a_1+a_2)^3\right] & w\in(a_1+a_2,m_1+a_2) \\ k_1\times\left[\frac{w}{2}(m_1-a_1)^2+\frac{m_1^2}{2}(a_1-a_2)-\frac{a_1^2a_2}{2}+a_1a_2m_1-\frac{a_1^3}{6}-\frac{m_1^3}{3}\right] & w\in(m_1+a_2,a_1+m_2) \\ k_1\times\left[-\frac{w^3}{6}+\frac{w^2}{2}(a_1+a_2)+w\left(\frac{m_1^2}{2}+\frac{m_2^2}{2}-a_1m_1-a_2m_2-a_1a_2\right)\right. & \\ \left.+\frac{m_1^2}{2}(a_1-a_2)-\frac{m_2^2}{2}(a_1-a_2)+a_1a_2(m_1+m_2)-\frac{m_1^3}{3}-\frac{m_2^3}{3}\right] & w\in(a_1+m_2,m_1+m_2) \end{cases} \tag{12}$$

Remark: For the remaining Cases II, III and IV the equations corresponding to eqs. (6) to (9) can be obtained on replacing $a_2\to b_2$, $a_1\to b_1$ and $a_1\to b_1$, $a_2\to b_2$ in these equations respectively.

Now following the lines of Case I and using the new equations obtained in the light of the above remark, we get the following components of $h(w)$ from $h_2(w)$ to $h_4(w)$ for the Cases II, III and IV.

Case II. $a_1\le x_1\le m_1$ and $m_2\le x_2\le b_2$

Here $k_2=\dfrac{4}{(b_1-a_1)(b_2-a_2)(m_1-a_1)(b_2-m_2)}$

If $a_1+b_2<m_1+m_2$:

$$h_2(w)=\begin{cases} k_2\times\left[-\frac{w^3}{6}+\frac{w^2}{2}(a_1+b_2)+w\left(-\frac{a_1^2}{2}+\frac{m_2^2}{2}-b_2m_2-a_1b_2\right)\right. & \\ \left.+\frac{m_2^2}{2}(b_2-a_1)+\frac{a_1^2b_2}{2}+a_1b_2m_2+\frac{a_1^3}{6}-\frac{m_2^3}{3}\right] & w\in(a_1+m_2,a_1+b_2) \\ k_2\times\left[\frac{w}{2}(b_2-m_2)^2+\frac{m_2^2}{2}(b_2-a_1)-\frac{a_1b_2^2}{2}+a_1b_2m_2-\frac{b_2^3}{6}-\frac{m_2^3}{3}\right] & w\in(a_1+b_2,m_1+m_2) \\ k_2\times\left[\frac{w^3}{6}-\frac{w^2}{2}(a_1+b_2)+w\left(-\frac{m_1^2}{2}+\frac{b_2^2}{2}+a_1m_1+a_1b_2\right)\right. & \\ \left.+\frac{m_1^2}{2}(b_2-a_1)-\frac{a_1b_2^2}{2}-a_1b_2m_1-\frac{b_2^3}{6}+\frac{m_1^3}{3}\right] & w\in(m_1+m_2,m_1+b_2) \end{cases} \tag{13}$$

If $a_1+b_2=m_1+m_2$:

$$h_2(w)=\begin{cases} k_2\times\left[-\dfrac{w^3}{6}+\dfrac{w^2}{2}(a_1+b_2)+w\left(-\dfrac{a_1^2}{2}+\dfrac{m_2^2}{2}-b_2m_2-a_1b_2\right)\right. \\ \left.\quad+\dfrac{m_2^2}{2}(b_2-a_1)+\dfrac{a_1^2b_2}{2}+a_1b_2m_2+\dfrac{a_1^3}{6}-\dfrac{m_2^3}{3}\right] & w\in(a_1+m_2,a_1+b_2) \\ k_2\times\left[\dfrac{w^3}{6}-\dfrac{w^2}{2}(a_1+b_2)+w\left(-\dfrac{m_1^2}{2}+\dfrac{b_2^2}{2}+a_1m_1+a_1b_2\right)\right. \\ \left.\quad+\dfrac{m_1^2}{2}(b_2-a_1)-\dfrac{a_1b_2^2}{2}-a_1b_2m_1-\dfrac{b_2^3}{6}+\dfrac{m_1^3}{3}\right] & w\in(a_1+b_2,m_1+b_2) \end{cases} \tag{14}$$

If $a_1+b_2>m_1+m_2$:

$$h_2(w)=\begin{cases} k_2\times\left[-\dfrac{w^3}{6}+\dfrac{w^2}{2}(a_1+b_2)+w\left(-\dfrac{a_1^2}{2}+\dfrac{m_2^2}{2}-b_2m_2-a_1b_2\right)\right. \\ \left.\quad+\dfrac{m_2^2}{2}(b_2-a_1)+\dfrac{a_1^2b_2}{2}+a_1b_2m_2+\dfrac{a_1^3}{6}-\dfrac{m_2^3}{3}\right] & w\in(a_1+m_2,m_1+m_2) \\ k_2\times\left[-\dfrac{w}{2}(m_1-a_1)^2+\dfrac{m_1^2}{2}(b_2-a_1)+\dfrac{a_1^2b_2}{2}-a_1b_2m_1+\dfrac{a_1^3}{6}+\dfrac{m_1^3}{3}\right] & w\in(m_1+m_2,a_1+b_2) \\ k_2\times\left[\dfrac{w^3}{6}-\dfrac{w^2}{2}(a_1+b_2)+w\left(-\dfrac{m_1^2}{2}+\dfrac{b_2^2}{2}+a_1m_1+a_1b_2\right)\right. \\ \left.\quad+\dfrac{m_1^2}{2}(b_2-a_1)-\dfrac{a_1b_2^2}{2}-a_1b_2m_1-\dfrac{b_2^3}{6}+\dfrac{m_1^3}{3}\right] & w\in(a_1+b_2,m_1+b_2) \end{cases} \tag{15}$$

Case III. $m_1\le x_1\le b_1$ and $a_2\le x_2\le m_2$

Here $k_3=\dfrac{4}{(b_1-a_1)(b_2-a_2)(b_1-m_1)(m_2-a_2)}$

If $m_1+m_2<b_1+a_2$:

$$h_3(w)=\begin{cases} k_3\times\left[-\dfrac{w^3}{6}+\dfrac{w^2}{2}(b_1+a_2)+w\left(\dfrac{m_1^2}{2}-\dfrac{a_2^2}{2}-b_1m_1-b_1a_2\right)\right. \\ \left.+\dfrac{m_1^2}{2}(b_1-a_2)+\dfrac{b_1a_2^2}{2}+b_1a_2m_1+\dfrac{a_2^3}{6}-\dfrac{m_1^3}{3}\right] & w\in(m_1+a_2,m_1+m_2) \\ k_3\times\left[-\dfrac{w}{2}(m_2-a_2)^2+\dfrac{m_2^2}{2}(b_1-a_2)+\dfrac{b_1a_2^2}{2}-b_1a_2m_2+\dfrac{a_2^3}{6}+\dfrac{m_2^3}{3}\right] & w\in(m_1+m_2,b_1+a_2) \\ k_3\times\left[\dfrac{w^3}{6}-\dfrac{w^2}{2}(b_1+a_2)+w\left(-\dfrac{m_2^2}{2}+\dfrac{b_1^2}{2}+a_2m_2+b_1a_2\right)\right. \\ \left.+\dfrac{m_2^2}{2}(b_1-a_2)-\dfrac{b_1^2a_2}{2}-b_1a_2m_2-\dfrac{b_1^3}{6}+\dfrac{m_2^3}{3}\right] & w\in(b_1+a_2,b_1+m_2) \end{cases} \quad (16)$$

If $m_1+m_2=b_1+a_2$:

$$h_3(w)=\begin{cases} k_3\times\left[-\dfrac{w^3}{6}+\dfrac{w^2}{2}(b_1+a_2)+w\left(\dfrac{m_1^2}{2}-\dfrac{a_2^2}{2}-b_1m_1-b_1a_2\right)\right. \\ \left.+\dfrac{m_1^2}{2}(b_1-a_2)+\dfrac{b_1a_2^2}{2}+b_1a_2m_1+\dfrac{a_2^3}{6}-\dfrac{m_1^3}{3}\right] & w\in(m_1+a_2,m_1+m_2) \\ k_3\times\left[\dfrac{w^3}{6}-\dfrac{w^2}{2}(b_1+a_2)+w\left(-\dfrac{m_2^2}{2}+\dfrac{b_1^2}{2}+a_2m_2+b_1a_2\right)\right. \\ \left.+\dfrac{m_2^2}{2}(b_1-a_2)-\dfrac{b_1^2a_2}{2}-b_1a_2m_2-\dfrac{b_1^3}{6}+\dfrac{m_2^3}{3}\right] & w\in(m_1+m_2,b_1+m_2) \end{cases} \quad (17)$$

If $m_1+m_2>b_1+a_2$:

$$h_3(w)=\begin{cases} k_3\times\left[-\dfrac{w^3}{6}+\dfrac{w^2}{2}(b_1+a_2)+w\left(\dfrac{m_1^2}{2}-\dfrac{a_2^2}{2}-b_1m_1-b_1a_2\right)\right. \\ \left.+\dfrac{m_1^2}{2}(b_1-a_2)+\dfrac{b_1a_2^2}{2}+b_1a_2m_1+\dfrac{a_2^3}{6}-\dfrac{m_1^3}{3}\right] & w\in(m_1+a_2,b_1+a_2) \\ k_3\times\left[-\dfrac{w}{2}(b_1-m_1)^2+\dfrac{m_1^2}{2}(b_1-a_2)-\dfrac{b_1^2a_2}{2}+b_1a_2m_1-\dfrac{b_1^3}{6}-\dfrac{m_1^3}{3}\right] & w\in(b_1+a_2,m_1+m_2) \\ k_3\times\left[\dfrac{w^3}{6}-\dfrac{w^2}{2}(b_1+a_2)+w\left(-\dfrac{m_2^2}{2}+\dfrac{b_1^2}{2}+a_2m_2+b_1a_2\right)\right. \\ \left.+\dfrac{m_2^2}{2}(b_1-a_2)-\dfrac{b_1^2a_2}{2}-b_1a_2m_2-\dfrac{b_1^3}{6}+\dfrac{m_2^3}{3}\right] & w\in(m_1+m_2,b_1+m_2) \end{cases} \quad (18)$$

Case IV. $m_1\le x_1\le b_1$ and $m_2\le x_2\le b_2$

Here $k_4=\dfrac{4}{(b_1-a_1)(b_2-a_2)(b_1-m_1)(b_2-m_2)}$

If $m_1 + b_2 < b_1 + m_2$:

$$h_4(w) = \begin{cases} k_4 \times \left[\frac{w^3}{6} - \frac{w^2}{2}(b_1+b_2) + w\left(-\frac{m_1^2}{2} - \frac{m_2^2}{2} + b_1 m_1 + b_2 m_2 + b_1 b_2 \right) \right. \\ \left. -\frac{m_1^2}{2}(b_1-b_2) + \frac{m_2^2}{2}(b_1-b_2) - a_1 a_2 (m_1+m_2) + \frac{m_1^3}{3} + \frac{m_2^3}{3} \right] & w \in (m_1+m_2, m_1+b_2) \\ k_4 \times \left[-\frac{w}{2}(b_2-m_2)^2 + \frac{m_2^2}{2}(b_1-b_2) + \frac{b_1 b_2^2}{2} - b_1 b_2 m_2 + \frac{b_2^3}{6} + \frac{m_2^3}{3} \right] & w \in (m_1+b_2, b_1+m_2) \\ k_4 \times \left[-\frac{w^3}{6} + \frac{w^2}{2}(b_1+b_2) - \frac{w}{2}(b_1+b_2)^2 + \frac{1}{6}(b_1+b_2)^3 \right] & w \in (b_1+m_2, b_1+b_2) \end{cases} \tag{19}$$

If $m_1 + b_2 = b_1 + m_2$:

$$h_4(w) = \begin{cases} k_4 \times \left[\frac{w^3}{6} - \frac{w^2}{2}(b_1+b_2) + w\left(-\frac{m_1^2}{2} - \frac{m_2^2}{2} + b_1 m_1 + b_2 m_2 + b_1 b_2 \right) \right. \\ \left. -\frac{m_1^2}{2}(b_1-b_2) + \frac{m_2^2}{2}(b_1-b_2) - a_1 a_2 (m_1+m_2) + \frac{m_1^3}{3} + \frac{m_2^3}{3} \right] & w \in (m_1+m_2, m_1+b_2) \\ k_4 \times \left[-\frac{w^3}{6} + \frac{w^2}{2}(b_1+b_2) - \frac{w}{2}(b_1+b_2)^2 + \frac{1}{6}(b_1+b_2)^3 \right] & w \in (m_1+b_2, b_1+b_2) \end{cases} \tag{20}$$

If $m_1 + b_2 > b_1 + m_2$:

$$h_4(w) = \begin{cases} k_4 \times \left[\frac{w^3}{6} - \frac{w^2}{2}(b_1+b_2) + w\left(-\frac{m_1^2}{2} - \frac{m_2^2}{2} + b_1 m_1 + b_2 m_2 + b_1 b_2 \right) \right. \\ \left. -\frac{m_1^2}{2}(b_1-b_2) + \frac{m_2^2}{2}(b_1-b_2) - a_1 a_2 (m_1+m_2) + \frac{m_1^3}{3} + \frac{m_2^3}{3} \right] & w \in (m_1+m_2, b_1+m_2) \\ k_4 \times \left[-\frac{w}{2}(b_1-m_1)^2 - \frac{m_1^2}{2}(m_1-m_2) + \frac{b_1^2 b_2}{2} - b_1 b_2 m_1 + \frac{b_1^3}{6} + \frac{m_1^3}{3} \right] & w \in (b_1+m_2, m_1+b_2) \\ k_4 \times \left[-\frac{w^3}{6} + \frac{w^2}{2}(b_1+b_2) - \frac{w}{2}(b_1+b_2)^2 + \frac{1}{6}(b_1+b_2)^3 \right] & w \in (m_1+b_2, b_1+b_2) \end{cases} \tag{21}$$

Illustration

We now find $h(w)$ in the case when $a_1 < a_2, m_1 < m_2, b_1 < b_2$ with $m_1 = a_2, b_1 = m_2$ and the conditions $m_1 - a_1 = 2(b_1 - m_1), m_2 - a_2 = b_2 - m_2$. For this, consideration of appropriate expressions of $h_1(w)$ to $h_4(w)$ are required which, in turn, depend upon the relative magnitudes of the quantities $a_1 + m_2$, $m_1 + a_2, a_1 + b_2, m_1 + m_2, b_1 + a_2, b_1 + m_2$ and $m_1 + b_2$ lying in the interval $[a_1 + a_2, b_1 + b_2]$ of the variable w. For our case, it is easy to observe that the sequence of these quantities will be $a_1 + a_2 < a_1 + m_2 < m_1 + a_2 = a_1 + b_2 < m_1 + m_2 = b_1 + a_2 < m_1 + b_2 = b_1 + m_2 < b_1 + b_2$ and hence the equations for this pdf will be (10), (13), (17) and (21). Combination of these equations according to the respective intervals give the pdf $h(w)$ on the support $[a_1 + a_2, b_1 + b_2]$ as shown below.

$$h(w)=k_1\times\left[\frac{w^3}{6}-\frac{w^2}{2}(a_1+a_2)+\frac{w}{2}(a_1+a_2)^2-\frac{1}{6}(a_1+a_2)^3\right]$$
$$w\in(a_1+a_2,a_1+m_2)$$

$$h(w)=k_1\times\left[\frac{w}{2}(m_2-a_2)^2-\frac{m_2^2}{2}(a_1-a_2)-\frac{a_1a_2^2}{2}+a_1a_2m_2-\frac{a_2^3}{6}-\frac{m_2^3}{3}\right]+k_2\times\left[-\frac{w^3}{6}+\frac{w^2}{2}(a_1+b_2)\right.$$
$$\left.+w\left(-\frac{a_1^2}{2}+\frac{m_2^2}{2}-b_2m_2-a_1b_2\right)+\frac{m_2^2}{2}(b_2-a_1)+\frac{a_1^2b_2}{2}+a_1b_2m_2+\frac{a_1^3}{6}-\frac{m_2^3}{3}\right]$$
$$w\in(a_1+m_2,m_1+a_2)$$

$$h(w)=k_1\times\left[-\frac{w^3}{6}+\frac{w^2}{2}(a_1+a_2)+w\left(\frac{m_1^2}{2}+\frac{m_2^2}{2}-a_1m_1-a_2m_2-a_1a_2\right)+\frac{m_1^2}{2}(a_1-a_2)-\frac{m_2^2}{2}(a_1-a_2)\right.$$
$$\left.+a_1a_2(m_1+m_2)-\frac{m_1^3}{3}-\frac{m_2^3}{3}\right]+K_2\times\left[\frac{w}{2}(b_2-m_2)^2+\frac{m_2^2}{2}(b_2-a_1)-\frac{a_1b_2^2}{2}+a_1b_2m_2-\frac{b_2^3}{6}-\frac{m_2^3}{3}\right]$$
$$+K_3\times\left[-\frac{w^3}{6}+\frac{w^2}{2}(b_1+a_2)+w\left(\frac{m_1^2}{2}-\frac{a_2^2}{2}-b_1m_1-b_1a_2\right)+\frac{m_1^2}{2}(b_1-a_2)+\frac{b_1a_2^2}{2}+b_1a_2m_1+\frac{a_2^3}{6}-\frac{m_1^3}{3}\right]$$
$$w\in(m_1+a_2,m_1+m_2)$$

$$h(w)=k_2\times\left[\frac{w^3}{6}-\frac{w^2}{2}(a_1+b_2)+w\left(-\frac{m_1^2}{2}+\frac{b_2^2}{2}+a_1m_1+a_1b_2\right)+\frac{m_1^2}{2}(b_2-a_1)-\frac{a_1b_2^2}{2}-a_1b_2m_1\right.$$
$$\left.-\frac{b_2^3}{6}+\frac{m_1^3}{3}\right]+k_3\times\left[\frac{w^3}{6}-\frac{w^2}{2}(b_1+a_2)+w\left(-\frac{m_2^2}{2}+\frac{b_1^2}{2}+a_2m_2+b_1a_2\right)+\frac{m_2^2}{2}(b_1-a_2)-\frac{b_1^2a_2}{2}\right.$$
$$\left.-b_1a_2m_2-\frac{b_1^3}{6}+\frac{m_2^3}{3}\right]+k_4\times\left[\frac{w^3}{6}-\frac{w^2}{2}(b_1+b_2)+w\left(-\frac{m_1^2}{2}-\frac{m_2^2}{2}+b_1m_1+b_2m_2+b_1b_2\right)\right.$$
$$\left.-\frac{m_1^2}{2}(b_1-b_2)+\frac{m_2^2}{2}(b_1-b_2)-a_1a_2(m_1+m_2)+\frac{m_1^3}{3}+\frac{m_2^3}{3}\right]$$
$$w\in(m_1+m_2,m_1+b_2)$$

$$h(w)=k_4\times\left[-\frac{w^3}{6}+\frac{w^2}{2}(b_1+b_2)-\frac{w}{2}(b_1+b_2)^2+\frac{1}{6}(b_1+b_2)^3\right]$$
$$w\in(m_1+b_2,b_1+b_2)$$

It can be verified that integrals of above values of $h(w)$, in the corresponding intervals, add up to 1 i.e., $\int_{a_1+a_2}^{b_1+b_2}h(w)dw=1$.

Fig. 1 shows the pdf $h(w)$ for the above illustrated case for selected values of a_1,a_2,b_1,b_2,m_1 and m_2.

Figs. 2, 3 and 4 show the variations in $h(w)$ with respect to variations in m_2,a_2 and b_2 respectively, keeping others parameters fixed.

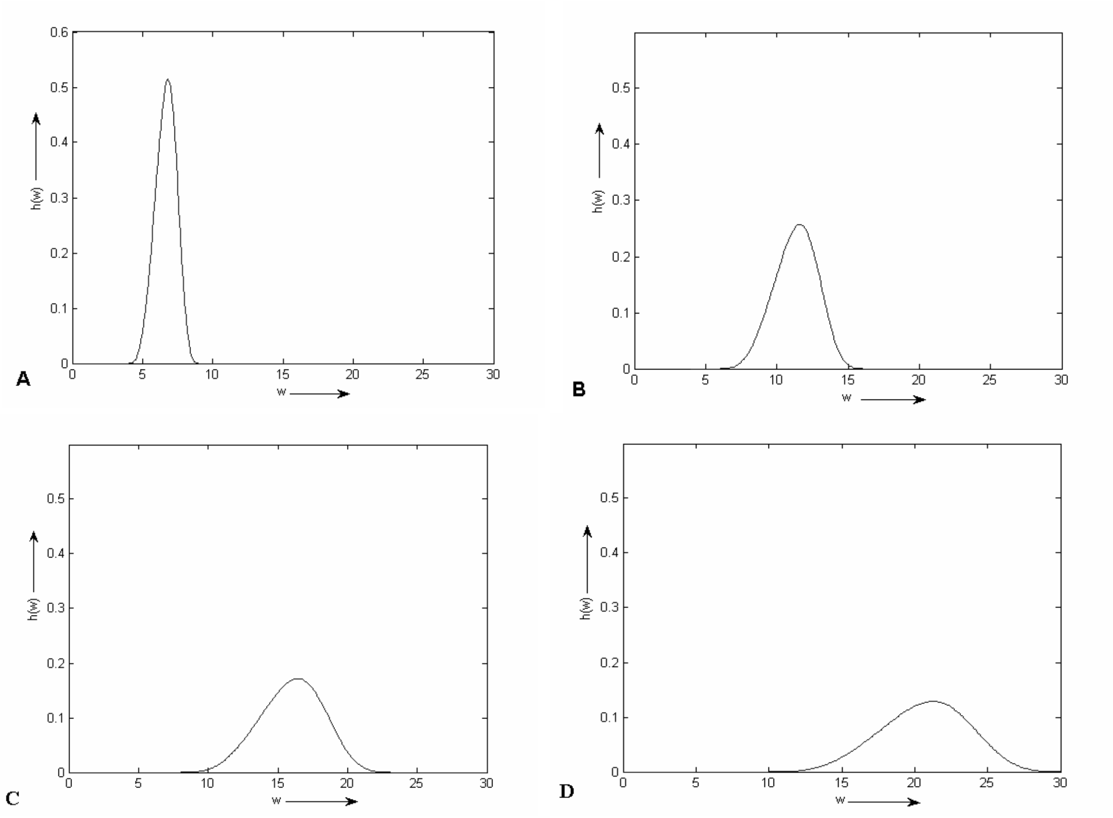

Fig. 1. $h(w)$ for selected values of a_1, a_2, b_1, b_2, m_1 and m_2 .

A: $a_1 = 1, m_1 = 3, b_1 = 4;\ a_2 = 3, m_2 = 4, b_2 = 5$

B: $a_1 = 1, m_1 = 5, b_1 = 7;\ a_2 = 5, m_2 = 7, b_2 = 9$

C: $a_1 = 1, m_1 = 7, b_1 = 10;\ a_2 = 7, m_2 = 10, b_2 = 13$

D: $a_1 = 1, m_1 = 9, b_1 = 13;\ a_2 = 9, m_2 = 13, b_2 = 17$

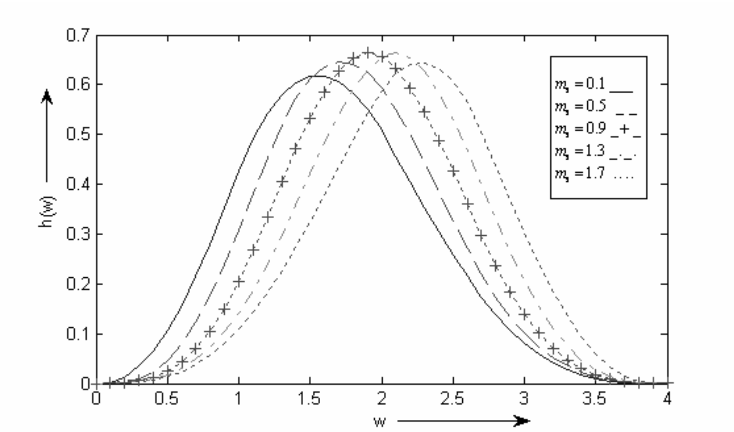

Fig. 2. $h(w)$ for different values of m_2 with $(a_1 = 0 = a_2, m_1 = 0.9, b_1 = 2 = b_2)$

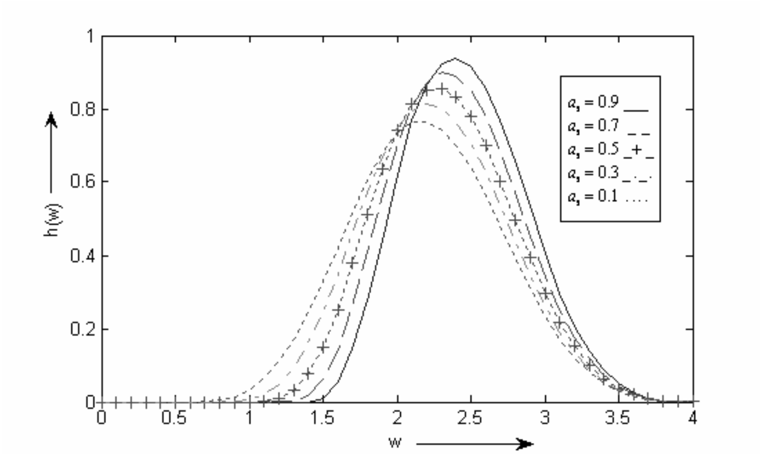

Fig. 3. $h(w)$ for different values of a_2 with $(a_1 = 0.5, m_1 = 1 = m_2, b_1 = 2 = b_2)$

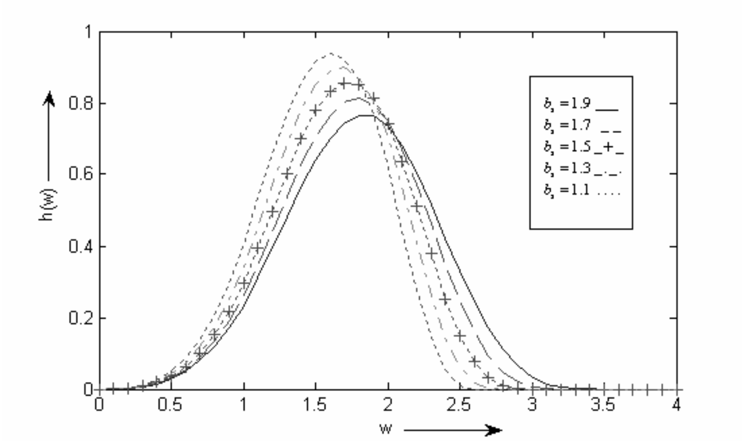

Fig. 4. $h(w)$ for different values of b_2 with $(a_1 = 0 = a_2, m_1 = 1 = m_2, b_1 = 1.5)$

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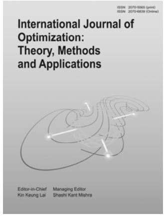

Duality for a Convex Fractional Programming under Fuzzy Environment

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Abstract. In this paper, we study a particular type of convex fractional programming problem and its dual under fuzzy environment. We present appropriate duality results for a fuzzy environment using aspiration level approach. This study use linear membership functions to represent fulfillment of the decision maker's degree of satisfaction.

Keywords: Fuzzy set theory, fractional programming, duality theory, primal-dual problems.

1 Introduction

Mathematical programming finds an extensive use in facilitating managerial decision situations in a large number of domains. An important class of mathematical programming problems is fractional programming which deals with situations where a ratio between physical and/or economical functions, is maximized (or minimized).

There are many decision situations that necessitate consideration of uncertainties in working environment best captured by fuzzy set theory. The concept of decision making in fuzzy environment was first proposed by Bellman and Zadeh [1]. Subsequently, Tanaka et al. [2] made use of this concept in mathematical programming. The use of fuzzy set theory in fractional programming has been discussed by many authors, e.g. Luhandjula [3], Dutta et al. [4], Ravi and Reddy [5], Gupta and Bhatia [6], Chakraborty and Gupta [7], Stancu-Minasian and Pop [8], Mehra et al. [9], Pop and Stancu-Minasian [10].

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The main focus in fuzzy fractional programming like other fuzzy mathematical programming problems has been placed on how to identify the best compromise solution yielding the highest possible degree of membership(overall satisfaction). It is natural therefore that the duality theory plays an important role for attaining the goal in a fuzzy environment as well. However, only few studies exploring duality in fuzzy fractional programming have appeared in literature. Lee et al. [11] developed a parallel algorithm and studied duality for a fuzzy multiobjective linear fractional programming problem. Wu [12] developed duality theory in fuzzy optimization problems formulated by the Wolfe's primal and dual pair. Gupta and Mehlawat [13] studied duality for fuzzy linear fractional programming under fuzzy environment using both linear and nonlinear membership functions.

In this paper, we attempt to generate duality results for a particular type of convex fractional programming problem and its dual under fuzzy environment using the aspiration level approach described by Zimmermann [14]. The aspiration level approach used in the present study is based on the fact that in practice a decision maker is more comfortable describing fuzzy constraints or establishing aspiration levels for the objective and/or constraints than specifying a large number of fuzzy numbers for the various parameters of the problem. Moreover, the aspiration level approach is more effective as it does not require consistency in the decision maker's judgment; the aspiration levels are more like a probe than weighting parameters.

The present research relies on the fact that the duality between fuzzy primal-dual pair can be relaxed (with additional terms) from the conventional duality (without additional terms) owing to the presence of fuzzy constraints (soft constraints) as developed by Bector and Chandra [15] for fuzzy primal-dual linear programming problems. However, the duality results generated for a fuzzy environment must conform to the corresponding results for the crisp situations. This implies that *the duality results for a fuzzy environment are a special case and lead to the corresponding results from crisp situations*. The concept of duality for a fuzzy environment used in the present study is well supported by a significant amount of prior research for fuzzy primal-dual linear programming problems, e.g. Hamacher et al. [16], Rödder and Zimmermann [17], Liu et al. [18], Bector and Chandra [15, 19], Bector et al. [20, 21], Vijay et al. [22].

This paper is organized as follows. In Section 2, we consider a pair of fuzzy primal-dual fractional programming problems in which vague aspiration levels are represented by linear membership functions. We establish appropriate duality results for a fuzzy environment. In Section 3, we present numerical illustrations of the duality results and related situations. Finally, some concluding remarks are made in Section 4.

2 Duality for a Fuzzy Convex Fractional Programming Problem

We consider the following convex fractional programming problem and its dual studied in [23].

$$\text{(PCFPP)} \min f(x) = \frac{(c^t x)^2}{d^t x}$$

subject to

$$Ax \geq b,$$
$$x \geq 0,$$

$$\text{(DCFPP)} \max g(u,v) = b^t u$$
$$\text{subject to}$$
$$A^t u + dv^2 \leq 2cv,$$
$$u, v \geq 0,$$

where A is an $m \times n$ matrix, x, c and d are column vectors with n components, b is a column vector with m components. Let $S = \{x \mid Ax \geq b, x \geq 0\}$ be the domain of feasible solutions for the primal problem (PCFPP). On domain S, assume that $c^t x \geq 0$ and $d^t x > 0$. Let $T = \{(u,v) \mid u \geq 0, v \geq 0, A^t u + dv^2 \leq 2cv\}$ be the domain of feasible solutions for the dual problem (DCFPP).

Consider following fuzzy versions of the (PCFPP) and the (DCFPP) in the sense of Zimmermann [14]. Let us call them (FPCFPP) and (FDCFPP).

$$\text{(FPCFPP)Find } x \in R^n \text{ subject to}$$
$$f(x) = \frac{(c^t x)^2}{d^t x} \lesssim z_0,$$
$$Ax \gtrsim b,$$
$$x \geq 0.$$

$$\text{(FDCFPP)Find } u \in R^m, v \in R \text{ subject to}$$
$$g(u,v) = b^t u \gtrsim w_0,$$
$$A^t u + dv^2 \lesssim 2cv,$$
$$u, v \geq 0.$$

Here "$\lesssim$" and "$\gtrsim$" indicates that the inequalities are flexible and may be described by a fuzzy set whose membership function represents fulfillment of the decision maker's degree of satisfaction and have interpretation of "essentially less than" and "essentially greater than" in the sense of Zimmermann [14]. Also z_0 and w_0 are aspiration levels of the two objectives.

Further, let p_0, $p_i (i = 1, 2, \cdots, m)$, be subjectively chosen constants of admissible violations associated with the objective function and the constraints of the problem (PCFPP).

Next, we define linear membership functions $\mu_i : R \to [0,1]$ to obtain a degree of satisfaction in the problem (FPCFPP).

$$\mu_0(x) = \begin{cases} 1 \text{ if } f(x) \leq z_0, \\ 1 - \dfrac{f(x) - z_0}{p_0} \text{ if } z_0 < f(x) \leq z_0 + p_0, \\ 0 \text{ if } f(x) > z_0 + p_0, \end{cases} \tag{1}$$

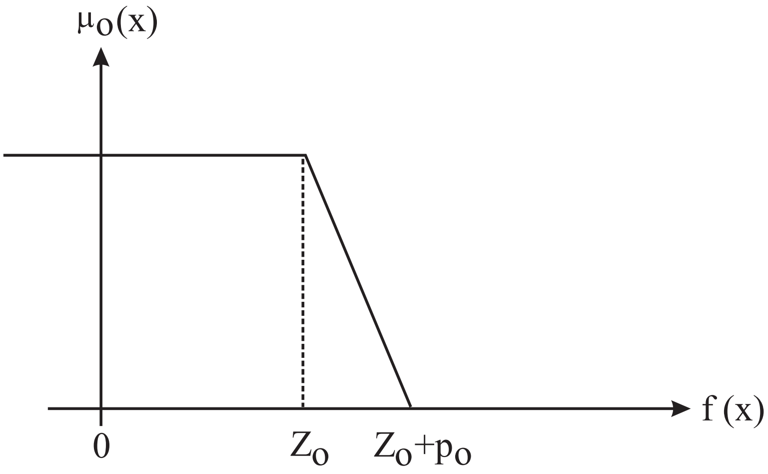

Fig. 1 Membership function for the objective function value goal

and

$$\mu_i(x) = \begin{cases} 1 \text{ if } A_{i\cdot}\, x \ge b_i, \\ 1 - \dfrac{b_i - A_{i\cdot}\, x}{p_i} \text{ if } b_i - p_i \le A_{i\cdot}\, x < b_i, \\ 0 \text{ if } A_{i\cdot}\, x < b_i - p_i. \end{cases} \tag{2}$$

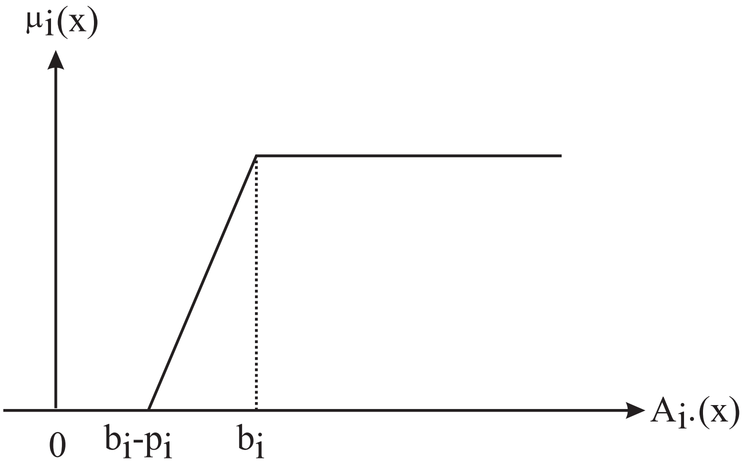

Fig. 2 Membership function for the fuzzy inequality constraint $A_{i\cdot}x - b_i \gtrsim 0$

Using the "min" operator to aggregate the overall satisfaction and following Zimmermann [14], the crisp equivalent of the problem (FPCFPP) is

(CP) $\min -\lambda$

subject to

$$(\lambda-1)p_0 \le z_0 - f(x), \tag{3}$$

$$(\lambda-1)p_i \le A_{i\cdot}\,x - b_i, \qquad (i=1,2,\cdots,m), \tag{4}$$

$$\lambda \le 1, \tag{5}$$

$$x, \lambda \ge 0, \tag{6}$$

where $A_{i\cdot}$ $(i=1,2,\cdots,m)$ denotes the i th row of the matrix A and b_i is the i th component of b.

Let $q_j\,(j=0,1,2,\cdots,n)$ be subjectively chosen constants of admissible violations of the objective and the constraint functions of the problem (DCFPP). On the same lines as above, we obtain crisp equivalent of the problem (FDCFPP) as

(CD) $\max \eta$

subject to

$$(\eta-1)q_0 \le g(u,v) - w_0, \tag{7}$$

$$(\eta-1)q_j \le 2c_j v - A_{j\cdot}^t\,u - d_j v^2, \quad (j=1,2,\cdots,n), \tag{8}$$

$$\eta \le 1, \tag{9}$$

$$u, v, \eta \ge 0. \tag{10}$$

Next, we establish appropriate duality results under fuzzy environment for the pair (CP), (CD).

Theorem 1. Let (x,λ) be feasible for (CP) and (u,v,η) be feasible for (CD). Then

$$(\eta-1)q^t x + (\lambda-1)p^t u \le f(x) - g(u,v). \tag{11}$$

Proof: Assume that $S \ne \phi$ and $T \ne \phi$. Multiply (8) by $x \ge 0$, we get

$$(\eta-1)q^t x \le 2x^t cv - x^t A^t u - x^t dv^2. \tag{12}$$

Also, multiply (4) by $u \ge 0$, we get

$$(\lambda-1)p^t u \le x^t A^t u - b^t u. \tag{13}$$

Adding (12) and (13), we get

$$(\eta-1)q^t x + (\lambda-1)p^t u \le 2x^t cv - x^t dv^2 - b^t u,$$

or

$$(\eta-1)q^t x + (\lambda-1)p^t u \le -b^t u - (v(d^t x)^{1/2} - (c^t x)(d^t x)^{-1/2})^2 + \frac{(c^t x)^2}{d^t x},$$

or

$$(\eta-1)q^t x+(\lambda-1)p^t u+b^t u+(v(d^t x)^{1/2}-(c^t x)(d^t x)^{-1/2})^2-\frac{(c^t x)^2}{d^t x}\leq 0.$$

Therefore,

$$(\eta-1)q^t x+(\lambda-1)p^t u+b^t u-\frac{(c^t x)^2}{d^t x}\leq 0,$$

or

$$(\eta-1)q^t x+(\lambda-1)p^t u\leq f(x)-g(u,v).$$

Hence the result.

The inequality (11) as defined above is a generalization of the crisp weak duality result, i.e. the crisp weak duality inequality $g(u,v)\leq f(x)$ is generalized to the fuzzy weak duality inequality $g(u, v)\lesssim f(x)$. In an economic analysis of the weak duality result, the fuzzy inequality assumes that the decision maker is able to maximize or minimize the "utility functions" and thus take into consideration all possible decisions of the competitors. Then, the decision maker can draw upon preference functions for his losses from a simultaneous consideration of the competitor's decision and his own. Thus, we obtain a special kind of bounded relationship between the solutions that satisfy the decision maker's assessments of prices and quantities in competitive situations, as opposed to a strict optimizer such as a maximizer or minimizer.

In order to satisfy the fuzzy inequality, some tolerance must be given for the satisfying solutions of the decision makers, which is, in general, not bounded. By relaxing the fuzzy inequality, we allow the decision makers to make decisions that maximize (or minimize) their utility functions, subject to the bounded relationship defined in the inequality (11).

Remark 1. It may be noted that for $\lambda=1$ and $\eta=1$ (i.e. when the original problems are crisp), the inequality (11) reduces to $g(u,v)\leq f(x)$, which is the standard weak duality result in the crisp duality theory. Also, for $0<\lambda<1$ and $0<\eta<1$, the situation remains fuzzy. For given tolerance levels $(p=p_1, p_2,\cdots,p_m)$ and $(q=q_1,q_2,\cdots,q_n)$, the situation is quantified in the following expression:

$$(\eta-1)q^t x+(\lambda-1)p^t u$$

Remark 2. In addition to inequality (11), adding (3) and (7), we can arrive at

$$(\lambda-1)p_0+(\eta-1)q_0\leq(g(u,v)-f(x))+(z_0-w_0). \tag{14}$$

It may also be noted that the inequality (14) relates the relative difference of the aspiration levels z_0 of $f(x)$, and w_0 of $g(u,v)$, respectively, in terms of their tolerance levels p_0 and q_0. If in addition to $\lambda=1$ and $\eta=1$, we also have $z_0-w_0=0$, then combining the inequalities (11) and (14) gives $f(x)=g(u,v)$, i.e. x and w are optimal for the crisp problems (PCFPP) and (DCFPP).

Since (CP) and (CD) are not dual in the conventional sense but only crisp equivalents of the fuzzy pair (FPCFPP) and (FDCFPP), there may not be a direct or converse duality between them. However, like the crisp convex fractional programming duality scenario, we can verify the extent to which the crisp equivalent (11) of the fuzzy weak duality inequality is satisfied as an equality.

Theorem 2. Let $(\bar{x},\bar{\lambda})$ be feasible for (CP), and let $(\bar{u},\bar{v},\bar{\eta})$ be feasible for (CD) such that

(i) $(\bar{\eta}-1)q^t\bar{x}+(\bar{\lambda}-1)p^t\bar{u}=f(\bar{x})-g(\bar{u},\bar{v})$

(ii) $(\bar{\eta}-1)q_0+(\bar{\lambda}-1)p_0=(g(\bar{u},\bar{v})-f(\bar{x}))+(z_0-w_0)$

(iii) The aspiration levels z_0 and w_0 satisfy $w_0-z_0\le 0$

then $(\bar{x},\bar{\lambda})$ is optimal to (CP) and $(\bar{u},\bar{v},\bar{\eta})$ is optimal to (CD).

Proof: Let (x,λ) be feasible for (CP) and (u,v,η) be feasible for (CD). Using theorem 1, we have

$$(\eta-1)q^tx+(\lambda-1)p^tu-(f(x)-g(u,v))\le 0. \tag{15}$$

From (i) we are given that

$$(\bar{\eta}-1)q^t\bar{x}+(\bar{\lambda}-1)p^t\bar{u}=f(\bar{x})-g(\bar{u},\bar{v}). \tag{16}$$

The relations (15) and (16) imply that for any feasible solution (x,λ) of (CP) and for any feasible solution (u,v,η) of (CD), we have

$$\begin{aligned}&(\eta-1)q^tx+(\lambda-1)p^tu-(f(x)-g(u,v))\\&\le(\bar{\eta}-1)q^t\bar{x}+(\bar{\lambda}-1)p^t\bar{u}-(f(\bar{x})-g(\bar{u},\bar{v})).\end{aligned}$$

This further implies that $(\bar{x},\bar{\lambda},\bar{u},\bar{v},\bar{\eta})$ is optimal to the following problem whose maximum value is zero,

$$\begin{aligned}&\max\left((\eta-1)q^tx+(\lambda-1)p^tu-(f(x)-g(u,v))\right)\\&\text{subject to}\\&(\lambda-1)p_0\le z_0-f(x),\\&(\lambda-1)p_i\le A_{i\cdot}x-b_i,\quad (i=1,2,\cdots,m),\\&(\eta-1)q_0\le g(u,v)-w_0,\\&(\eta-1)q_j\le 2c_jv-A_{j\cdot}^t u-d_jv^2,\quad (j=1,2,\cdots,n),\\&\lambda,\eta\le 1,\\&x,u,v\ge 0,\\&\lambda,\eta\ge 0.\end{aligned}$$

Now from the given condition (i), we have

$$(\bar{\eta}-1)q^t\bar{x}+(\bar{\lambda}-1)p^t\bar{u}-(f(\bar{x})-g(\bar{u},\bar{v}))=0. \tag{17}$$

Also from given condition (ii), we have

$$(\bar{\eta}-1)q_0+(\bar{\lambda}-1)p_0-(g(\bar{u},\bar{v})-f(\bar{x}))-(z_0-w_0)=0. \tag{18}$$

Adding (17) and (18), we get

$$(\bar{\eta}-1)q^t\bar{x}+(\bar{\lambda}-1)p^t\bar{u}+(\bar{\lambda}-1)p_0+(\bar{\eta}-1)q_0+(w_0-z_0)=0.$$

Since each term in the above sum is non-positive (because $\bar{\lambda}, \bar{\eta} \le 1$) and, therefore, each of these terms should each be equal to zero, i.e.

$$(\bar{\eta}-1)q^t\bar{x}=0,$$
$$(\bar{\lambda}-1)p^t\bar{u}=0,$$
$$(\bar{\lambda}-1)p_0=0,$$
$$(\bar{\eta}-1)q_0=0,$$
$$w_0-z_0=0.$$

Since $(\lambda-1)p_0 \le 0$ and $(\eta-1)q_0 \le 0$ (because $\lambda, \eta \le 1$), we get

$$(\lambda-1)p_0 \le (\bar{\lambda}-1)p_0,$$

and

$$(\eta-1)q_0 \le (\bar{\eta}-1)q_0.$$

But $p_0 > 0$ and $q_0 > 0$, so canceling p_0 and q_0, we get $-\lambda \ge -\bar{\lambda}$ (or $\lambda \le \bar{\lambda}$) and $\eta \le \bar{\eta}$.

Hence the result.

In the present paper, since we are studying duality in a fuzzy environment, even if there is a strong duality between the fuzzy primal-dual pair (CP) and (CD) in the sense of Theorem 2, strong duality will be achieved only in the fuzzy sense as $g(u,v) \simeq f(x)$.

Remark 3. In crisp situations (i.e. when $\lambda=1$ and $\eta=1$), we get the following expression: $(\eta-1)q^t x+(\lambda-1)p^t u=0$. Following (i), (ii), and (iii) of Theorem 2, we obtain $g(u,v)=f(x)$, i.e. strong duality in the crisp sense.

3 Numerical Illustration

Consider the following pair of primal-dual fractional programming problems.

$$(\text{PCFPP}) \min f(x)=\frac{(2x_1+x_2)^2}{x_1+2x_2}$$

subject to

$$2x_1+x_2 \ge 6,$$
$$x_1+3x_2 \ge 8,$$
$$x_1, x_2 \ge 0,$$

$$(\text{DCFPP}) \max g(u,v)=6u_1+8u_2$$

subject to

$$2u_1+u_2+v^2-4v \le 0,$$
$$u_1+3u_2+2v^2-2v \le 0,$$
$$u_1, u_2, v \ge 0.$$

Take $p_0 = 2$, $p_1 = 1$, $p_2 = 2$, and $z_0 = 1$ for (PCFPP), the corresponding problem (CP) is

$$\begin{aligned}
&\min -\lambda \\
&\text{subject to} \\
&2\lambda x_1 + 4\lambda x_2 - 3x_1 - 6x_2 + 4x_1^2 + x_2^2 + 4x_1x_2 \le 0, \\
&-2x_1 - x_2 + \lambda \le -5, \\
&-x_1 - 3x_2 + 2\lambda \le -6, \\
&\lambda \le 1, \\
&x_1, x_2, \lambda \ge 0.
\end{aligned}$$

The optimal solution of (CP) is at $x_1^* = 0$, $x_2^* = 5.2$, $\lambda^* = 0.20$ and the optimal value of (CP) is $-\lambda^* = -0.20$.

Now taking $q_0 = 1$, $q_1 = 1$, $q_2 = 2$ and $w_0 = 1$ for (DCFPP), the corresponding problem (CD) becomes

$$\begin{aligned}
&\max \eta \\
&\text{subject to} \\
&-6u_1 - 8u_2 + \eta \le 0, \\
&2u_1 + u_2 + v^2 - 4v + \eta \le 1, \\
&u_1 + 3u_2 + 2v^2 - 2v + 2\eta \le 2, \\
&\eta \le 1 \\
&u_1, u_2, v, \eta \ge 0.
\end{aligned}$$

The optimal solution of (CD) is at $\eta^* = 1$, $u_1^* = 0.40$, $u_2^* = 0$, $v^* = 0.72$ and the optimal value of (CD) is $\eta^* = 1$.

For these optimal solutions, both the inequalities (11) and (14) are satisfied.

Since we are using an aspiration level approach in the present study, it is unreasonable for a decision maker to choose aspiration levels that are too aggressive or too weak. If the aspiration levels are too aggressive, then there are no achievable solutions and the decision maker is asked to relax the aspiration levels. On the other hand, if the aspiration levels are too weak, then too many (perhaps all) solutions may be identified as feasible. Hence, when using the satisfying approach, the decision maker should choose his aspiration levels tightly.

The problems (CP) and (CD) are nonlinear programming problems. Although for medium or large-sized problems one should expect that solving these nonlinear programming problems could be computationally difficult, this is not the case. Many excellent software systems for solving nonlinear programs are available. Some of them are based on Generalized Reduced Gradient Method developed by Lasdon and Waren [24], GRG2 [25], which can handle very efficiently large scale problems. We have used LINGO [26] that is an interactive interface to GRG2 to solve problems (CP) and (CD).

4 Conclusions

We limited our construction of primal-dual problems under fuzzy environment using linear membership functions in the present paper. Different crisp equivalents of the fuzzy primal-dual problems can be obtained using other types of membership functions on the basis of the decision maker's preferences. It would be interesting to explore the possibility of establishing duality results under fuzzy environment for such pairs of fuzzy primal-dual problems. The results obtained under fuzzy environment for the dual examined in this paper may not be tenable for other types of dual for the fractional programming problem under consideration. Thus, the choice of dual plays a significant role in the theory development and hence is a major contribution in the present study.

The crisp equivalents obtained in the present paper are nonlinear (even nonconvex) problems, where non linearity exists in the constraints. Software LINGO [26] has been used to solve the numerical illustrations. The crisp equivalents can also be solved using fuzzy decisive set method [27] and the modified subgradient method [28].

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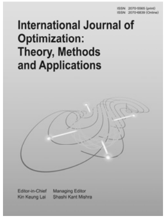

Second Order Duality for the Control Problems under $\rho-(\eta,\xi,\theta)-$invexity

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Abstract. In this paper, we introduce the concept of second order duality for the control problems using $\rho-(\eta,\xi,\theta)-$ invexity type conditions. Weak duality, strong duality and converse duality results of Mangasarian and Mond-Weir type of control problems are established under $\rho-(\eta,\xi,\theta)-$ invexity assumptions.

Keywords: second order duality control problem, $\rho-(\eta,\xi,\theta)-$invexity, weak duality, strong duality, converse duality, Mangasarian and Mond-Wier type duality.

1 Introduction

The study of second order duality is useful due to the computational advantage over first order duality as it gives bounds for the value of the objective function when approximations are used (see [7], [9], [10]). Many researchers [1, 4, 6, 13] have discussed various properties, extensions, and applications of generalized invex functions, for example in [17], Zalmai talked about $\rho-(\eta,\theta)-$invex functions. A number of duality theorems for the control problems have appeared in the literature (see, for example, [5], [8], [14], [15]). Mond and Hanson [11] discussed the dual for a class of variational problems with differential inequality constraints and established some duality results. Mond and Smart [12] extended the results of Mond and Hanson [11] for the control problems under invexity assumptions.

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In this paper we discuss the second order duality for the control problems under $\rho-(\eta,\xi,\theta)-$ invexity assumption and also study some duality results that is (weak, strong, converse). We give many examples and counterexamples to justify our work.

2 Notation and Preliminaries

Let $I=[a,b]$ be an interval (through out), $f: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, $g: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^r$ and $g: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^s$ be twice continuously differentiable functions. Consider the function $f(t, x(t),\dot{x}(t),u(t))$, where $x: I\to\mathbb{R}^n$ and $\dot{x}$ denotes the derivative of x with respect to t. u, λ, and μ are, respectively, $m-$, $r-$ and $s-$dimensional functions of t. Here t is an independent variable. All vectors will be taken as column vectors. The symbol z^T denotes for the transpose of a vector z. Denote the first partial derivatives of f with respect to $x(t)$ and $\dot{x}(t)$ by f_x and $f_{\dot{x}}$, respectively, that is,

$$f_x=\begin{pmatrix}\frac{\partial f}{\partial x_1}\\ \frac{\partial f}{\partial x_2}\\ \vdots\\ \frac{\partial f}{\partial x_n}\end{pmatrix},\qquad f_{\dot{x}}=\begin{pmatrix}\frac{\partial f}{\partial \dot{x}_1}\\ \frac{\partial f}{\partial \dot{x}_2}\\ \vdots\\ \frac{\partial f}{\partial \dot{x}_n}\end{pmatrix}.$$

Denote f_{xx} the Hessian matrix of f with respect to $x(t)$. Again g_x the $r\times n$, and h_x the $s\times n$ Jacobian matrix with respect to x. Similarly $f_u, f_{\dot{u}}, f_{\dot{u}u}, f_{u\dot{u}}, f_{\dot{x}u}, f_{u\dot{x}}, f_{\dot{x}x}, f_{x\dot{x}}; g_u, g_{\dot{u}}, g_{\dot{u}u}, g_{u\dot{u}}, g_{\dot{x}u}, g_{u\dot{x}}, g_{\dot{x}}, g_{\dot{x}x}, g_{x\dot{x}}$ and $h_u, h_{\dot{u}}, h_{\dot{u}u}, h_{u\dot{u}}, h_{\dot{x}u}, h_{u\dot{x}}, h_x, h_{\dot{x}}, h_{\dot{x}x}, h_{x\dot{x}}$ are also defined. In each Examples n and m are taken as 1 and $[0,1]^6=[0,1]\times[0,1]\times[0,1]\times[0,1]\times[0,1]\times[0,1]$.

Definition 1. The scalar functional $H(x,\dot{x},u)=\int_a^b h(t,x(t),\dot{x}(t),u(t))dt$ is said to be $p-(\eta,\xi,\theta)-$ invex in x, $\dot{x}$ and u with respect to the functions $\eta,\theta: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^n$ with $\eta=0$ at $t=a$ and $t=b$, and $\xi: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m$ with $\rho\in\mathbb{R}$ such that

$$H(x,\dot{x},u)-H(y,\dot{y},v)\geq\int_a^b\Big\{\eta^T h_x(t,y(t),\dot{y}(t),u(t))+\left(\frac{d}{dt}\eta^T\right)h_{\dot{x}}(t,y(t),\dot{y}(t),v(t))$$
$$+\xi^T h_u(t,y(t),\dot{y}(t),v(t))+\rho\,\|\,\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\,\|^2\Big\}dt.$$

Here $\|z\|=\left(\sum_{i=1}^n z_j^2\right)^{\frac{1}{2}}$ is the standard 2-norm of a vector z, z_i is the ith component of the vector z.

It follows that every invex function is $\rho-(\eta,\xi,\theta)-$ invex but the converse is not true from the following counterexample (1).

Example 1. Let $f: I\times[0,1]\times[0,1]\times[0,1]\to\mathbb{R}$ be defined by

$$f(t,x(t),\dot{x}(t),u(t)) = \frac{-x^2(t)u^2(t)t}{b^2-a^2}$$

The function $\int_a^b f(t,.,.,.)dt$ is not invex with respect to any $\eta : I\times[0,1]^6 \to \mathbb{R}$ and $\theta : I\times[0,1]^6 \to \mathbb{R}$ at $y(t)=0$ or $v(t)=0$. But $\int_a^b f(t,.,.,.)dt$ is $\rho-(\eta,\xi,\theta)-$ invex function for $\rho\in\mathbb{R}_-$, $y(t)v(t) > x(t)u(t)$, and any positive ξ and η.

Definition 2. The scalar functional $H(x,\dot{x},u)=\int_a^b h(t,x(t),\dot{x}(t),u(t))dt$ is said to be $\rho-(\eta,\xi,\theta)-$ pseudo-invex in x, $\dot{x}$ and u with respect to the functions $\eta,\theta : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}^n$ with $\eta=0$ at $t=a$ and $t=b$, and $\xi : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}^m$ with $\rho\in\mathbb{R}$ such that

$$\int_a^b \Big\{\eta^T h_x(t,y(t),\dot{y}(t),v(t)) + \left(\frac{d}{dt}\eta^T\right) h_{\dot{x}}(t,y(t),\dot{y}(t),v(t))$$
$$+\xi^T h_u(t,y(t),\dot{y}(t),v(t)) + \rho\,\|\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\|^2\Big\}dt \geq 0.$$
$$\Rightarrow H(x,\dot{x},u) \geq H(y,\dot{y},v)$$

It is noted that $\rho-(\eta,\xi,\theta)-$ invex function is $\rho-(\eta,\xi,\theta)-$ pseudo-invex but the converse is not true, follows from the following counterexample (2).

Example 2. Let $f : I\times[0,1]\times[0,1]\times[0,1] \to \mathbb{R}$ be defined by

$$f(t,x(t),\dot{x}(t)) = -\frac{4}{3}x^3(t)u^3(t)$$

Let the functions $\eta : I\times[0,1]^6 \to \mathbb{R}$, $\xi : I\times[0,1]^6 \to \mathbb{R}$ and $\theta : I\times[0,1]^6 \to \mathbb{R}$ be given by

$$\eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t)) = \begin{cases} y(t)(x(t)u(t)-y(t)v(t)), & \text{if } y(t)v(t) > x(t)u(t) \\ 0, & \text{if } y(t)v(t) \leq x(t)u(t) \end{cases}$$

$$\xi(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t)) = \begin{cases} v(t)(x(t)u(t)-y(t)v(t)), & \text{if } y(t)v(t) > x(t)u(t) \\ 0, & \text{if } y(t)v(t) \leq x(t)u(t) \end{cases}$$

$$\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t)) = \begin{cases} \sqrt{y^3(t)v^3(t)(y(t)v(t)-x(t)u(t))}, & \text{if } y(t)v(t) > x(t)u(t) \\ 0, & \text{if } y(t)v(t) \leq x(t)u(t) \end{cases}$$

To show that $\int_a^b f(t,.,.,.)dt$ is $\rho-(\eta,\xi,\theta)-$ pseudo-invex we have to verify the following

$$\int_a^b [\eta^T f_x(t,y(t),\dot{y}(t),v(t)) + \left(\frac{d}{dt}\eta^T\right) f_{\dot{x}}(t,y(t),\dot{y}(t),v(t)) + \xi^T f_u(t,y(t),\dot{y}(t),v(t))$$
$$+\rho\,\|\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\|^2]dt \geq 0$$

$$\Rightarrow \int_a^b f(t,x(t),\dot{x}(t),u(t))dt \geq \int_a^b f(t,y(t),\dot{y}(t),v(t))dt.$$

Taking $\rho=-1$.

$$\int_a^b [\eta^T f_x(t,y(t),\dot{y}(t),v(t))+\left(\frac{d}{dt}\eta^T\right)f_{\dot{x}}(t,y(t),\dot{y}(t),v(t))+\eta^T f_x(t,y(t),\dot{y}(t),v(t))$$
$$+\rho\,\|\,\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\,\|^2]dt$$
$$=\int_a^b 7y^3(t)v^3(t)(y(t)v(t)-x(t)u(t))dt$$
$$>0,\quad \text{for } y(t)v(t)>x(t)u(t).$$

Now $\int_a^b f(t,x(t),\dot{x}(t),u(t))dt-\int_a^b f(t,y(t),\dot{y}(t),v(t))dt$

$$=\int_a^b \frac{4}{3}(y^3(t)v^3(t)-x^3(t)u^3(t)dt$$
$$>0,\quad \text{for } y(t)v(t)>x(t)u(t).$$

Therefore our function $\int_a^b f(t,.,.,.)dt$ is not $\rho-(\eta,\xi,\theta)-$ invex but is $\rho-(\eta,\xi,\theta)-$ pseudo-invex for $y(t)v(t)>x(t)u(t)$.

Definition 3. The scalar functional $H(x,\dot{x},u)=\int_a^b h(t,x(t),\dot{x}(t),u(t))dt$ is said to be $\rho-(\eta,\xi,\theta)-$ quasi-invex in $x,\dot{x}$ and u with respect to the functions $\eta,\theta: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^n$ with $\eta=0$ at $t=a$ and $t=b$, and $\xi: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m$ with $\rho\in\mathbb{R}$ such that

$$H(x,\dot{x},u)\le H(y,\dot{y},v)$$
$$\Rightarrow\int_a^b\left\{\eta^T h_x(t,y(t),\dot{y}(t),v(t))+\left(\frac{d}{dt}\eta^T\right)h_{\dot{x}}(t,y(t),\dot{y}(t),v(t))\right.$$
$$\left.+\xi^T h_u(t,y(t),\dot{y}(t),v(t))+\rho\,\|\,\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\,\|^2\right\}dt\le 0.$$

It follows that every $\rho-(\eta,\xi,\theta)-$ invex function is $\rho-(\eta,\xi,\theta)-$ quasi-invex but the converse is not true, follows from the following counterexample (3).

Example 3. Let $f: I\times[0,1]\times[0,1]\times[0,1]\to\mathbb{R}$ be given by

$$f(t,x(t),\dot{x}(t),u(t))=-x^2(t)u^2(t)$$

Let the functions $\eta: I\times[0,1]^6\to\mathbb{R}$, $\xi: I\times[0,1]^6\to\mathbb{R}$ and $\theta: I\times[0,1]^6\to\mathbb{R}$ be given by

$$\eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))=\begin{cases} y(t)(y(t)v(t)-x(t)u(t)), & \text{if } x(t)u(t)>y(t)v(t)\\ 0, & \text{if } x(t)u(t)\le y(t)v(t)\end{cases}$$

$$\xi(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))=\begin{cases} v(t)(y(t)v(t)-x(t)u(t)), & \text{if } x(t)u(t)>y(t)v(t)\\ 0, & \text{if } x(t)u(t)\le y(t)v(t)\end{cases}$$

$$\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))=\begin{cases} \sqrt{y^2(t)v^2(t)(y(t)v(t)-x(t)u(t))}, & \text{if } x(t)u(t)>y(t)v(t)\\ 0, & \text{if } x(t)u(t)\le y(t)v(t)\end{cases}$$

Now

$$\int_a^b f(t,x(t),\dot{x}(t),u(t))dt - \int_a^b f(t,y(t),\dot{y}(t),v(t))dt - \Big[\int_a^b \Big\{\eta^T f_x(t,y(t),\dot{y}(t),v(t))$$
$$+\xi^T f_u(t,y(t),\dot{y}(t),v(t)) + \left(\frac{d}{dt}\eta^T\right) f_{\dot{x}}(t,y(t),\dot{y}(t),v(t))$$
$$+\rho \| \theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t)) \|^2 \Big\} dt\Big]$$
$$= \int_a^b \{y^2(t)v^2(t) - x^2(t)u^2(t) + 4y^2(t)v^2(t)(x(t)u(t) - y(t)v(t))$$
$$-\rho y^2(t)v^2(t)(x(t)u(t) - y(t)v(t))\}dt$$
$$< 0, \quad \text{for } \rho = -4, \text{ and } x(t)u(t) > y(t)v(t).$$

Hence the function $\int_a^b f(t,.,.,.)dt$ is not $\rho-(\eta,\xi,\theta)-$ invex, for $\rho=-4$, and $x(t)u(t) > y(t)v(t)$. But

$$\int_a^b f(t,x(t),\dot{x}(t),u(t))dt - \int_a^b f(t,y(t),\dot{y}(t),v(t))dt$$
$$= \int_a^b (y^2(t)v^2(t) - x^2(t)u^2(t))dt$$
$$< 0, \quad \text{for } x(t)u(t) > y(t)v(t),$$

and

$$\int_a^b \Big\{\eta^T f_x(t,y(t),\dot{y}(t),u(t)) + \left(\frac{d}{dt}\eta^T\right) f_{\dot{x}}(t,y(t),\dot{y}(t),v(t))$$
$$f_u(t,y(t),\dot{y}(t),v(t)) + \rho \| \theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t)) \|^2 \Big\}$$
$$= \int_a^b \Big\{-2y^2(t)v^2(t)(y(t)v(t) - x(t)u(t)) + \rho y^2(t)v^2(t)(y(t)v(t) - x(t)u(t))\Big\} dt$$
$$< 0, \quad \text{for } \rho = -4, \text{ and } x(t)u(t) > y(t)v(t).$$

Therefore the function $\int_a^b f(t,.,.,.)dt$ is $\rho-(\eta,\xi,\theta)-$ quasi-invex, but not $\rho-(\eta,\xi,\theta)-$ invex.

Consider the control primal problem

$$\text{(CP)} \min \int_a^b f(t,x(t),\dot{x}(t),u(t))dt,$$

$$\text{subject to } g(t,x(t),\dot{x}(t),u(t)) \le 0, \tag{1}$$

$$h(t,x(t),\dot{x}(t),u(t)) = 0, \tag{2}$$

$$x(a) = \gamma_1, \ \ x(b) = \gamma_2 \tag{3}$$

where *f*, *g* and *h* are twice continuously differentiable functions from $I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}$, $\mathbb{R}^r$ and $\mathbb{R}^s$, respectively.

Mangasarian Type Duality

Mangasarian's second order dual (*MCD*) of the control primal problem (*CP*) is given by

$$
\begin{aligned}
&\text{(MCD) } \max \int_a^b \{ f(t,y(t),\dot{y}(t),v(t)) + \lambda(t)^T g(t,y(t),\dot{y}(t),v(t)) \\
&+\mu(t)^T h(t,y(t),\dot{y}(t),v(t)) - \frac{1}{2}\beta(t)^T [f_{xx}(t,y(t),\dot{y}(t),v(t)) \\
&+\left(g_x(t,y(t),\dot{y}(t),v(t))^T \lambda(t)\right)_x + \left(h_x(t,y(t),\dot{y}(t),v(t))^T \mu(t)\right)_x \\
&-2\frac{d}{dt}(f_{x\dot{x}}(t,y(t),\dot{y}(t),v(t)) + \left(g_x(t,y(t),\dot{y}(t),v(t))^T \lambda(t)\right)_{\dot{x}} \\
&+\left(h_x(t,y(t),\dot{y}(t),v(t))^T \mu(t)\right)_{\dot{x}}) + \frac{d^2}{dt^2}(f_{\dot{x}\dot{x}}(t,y(t),\dot{y}(t),v(t)) \\
&+\left(g_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T \lambda(t)\right)_{\dot{x}} + \left(h_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T \mu(t)\right)_{\dot{x}})]\beta(t) \\
&-\frac{1}{2}\gamma(t)^T [f_{uu}(t,y(t),\dot{y}(t),v(t)) + \left(g_u(t,y(t),\dot{y}(t),v(t))^T \lambda(t)\right)_u \\
&+\left(h_u(t,y(t),\dot{y}(t),v(t))^T \mu(t)\right)_u + 2\{ f_{xu}(t,y(t),\dot{y}(t),v(t)) \\
&+\left(g_x(t,y(t),\dot{y}(t),v(t))^T \lambda(t)\right)_u + \left(h_x(t,y(t),\dot{y}(t),v(t))^T \mu(t)\right)_u \}]\gamma(t) \} dt
\end{aligned}
$$

subject to

$$
\begin{aligned}
&f(t,y(t),\dot{y}(t),v(t)) + g_x(t,y(t),\dot{y}(t),v(t))^T \lambda(t) \\
&+h_x(t,y(t),\dot{y}(t),v(t))^T \mu(t) - \frac{d}{dt}(f_{\dot{x}}(t,y(t),\dot{y}(t),v(t)) \\
&+g_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T \lambda(t) + h_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T \mu(t) \\
&+[f_{xx}(t,y(t),\dot{y}(t),v(t)) + \left(g_x(t,y(t),\dot{y}(t),v(t))^T \lambda(t)\right)_x \\
&+\left(h_x(t,y(t),\dot{y}(t),v(t))^T \mu(t)\right)_x - 2\frac{d}{dt}(f_{x\dot{x}}(t,y(t),\dot{y}(t),v(t)) \\
&+\left(g_x(t,y(t),\dot{y}(t),v(t))^T \lambda(t)\right)_{\dot{x}} + \left(h_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T \mu(t)\right)_{\dot{x}}) \\
&+\frac{d^2}{dt^2}(f_{\dot{x}\dot{x}}(t,y(t),\dot{y}(t),v(t)) + \left(g_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T \lambda(t)\right)_{\dot{x}} \\
&+\left(h_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T \mu(t)\right)_{\dot{x}})]\beta(t) = 0
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
&f_u(t,y(t),\dot{y}(t),v(t)) + g_u(t,y(t),\dot{y}(t),v(t))^T \lambda(t) \\
&+h_u(t,y(t),\dot{y}(t),v(t))^T \mu(t) + [f_{uu}(t,y(t),\dot{y}(t),v(t)) \\
&+\left(g_u(t,y(t),\dot{y}(t),v(t))^T \lambda(t)\right)_u + \left(h_u(t,y(t),\dot{y}(t),v(t))^T \mu(t)\right)_u
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
&+2\{ f_{xu}(t,y(t),\dot{y}(t),v(t)) + \left(g_x(t,y(t),\dot{y}(t),v(t))^T \lambda(t)\right)_u \\
&+\left(h_x(t,y(t),\dot{y}(t),v(t))^T \mu(t)\right)_u \}]\gamma(t)
\end{aligned} \tag{6}
$$

$$y(a)=\gamma_1,\ y(b)=\gamma_2 \tag{7}$$

$$\lambda(t)\in\mathbb{R}^r_+,\ \mu(t)\in\mathbb{R}^s,\ \beta(t)\in\mathbb{R}^n,\ \gamma(t)\in\mathbb{R}^m,\ t\in I. \tag{8}$$

Let

$$\begin{aligned}
&H_1(t,y(t),\dot y(t),v(t),\lambda(t),\mu(t)=f_{xx}(t,y(t),\dot y(t),v(t))\\
&+\left(g_x(t,y(t),\dot y(t),v(t))^T\lambda(t)\right)_x+\left(h_x(t,y(t),\dot y(t),v(t))^T\mu(t)\right)_x\\
&-2\frac{d}{dt}\left(f_{x\dot x}(t,y(t),\dot y(t),v(t))+\left(g_x(t,y(t),\dot y(t),v(t))^T\lambda(t)\right)_{\dot x}\right.\\
&\left.+\left(h_x(t,y(t),\dot y(t),v(t))^T\mu(t)\right)_{\dot x}\right)+\frac{d^2}{dt^2}\left(f_{\dot x\dot x}(t,y(t),\dot y(t),v(t))\right.\\
&\left.+\left(g_{\dot x}(t,y(t),\dot y(t),v(t))^T\lambda(t)\right)_{\dot x}+\left(h_{\dot x}(t,y(t),\dot y(t),v(t))^T\mu(t)\right)_{\dot x}\right),
\end{aligned}$$

and

$$\begin{aligned}
&H_2(t,y(t),\dot y(t),v(t),\lambda(t),\mu(t))\\
&=[f_{uu}(t,y(t),\dot y(t),v(t))+\left(g_u(t,y(t),\dot y(t),v(t))^T\lambda(t)\right)_u\\
&+\left(h_u(t,y(t),\dot y(t),v(t))^T\mu(t)\right)_u+2\left(f_{xu}(t,y(t),\dot y(t),v(t))\right.\\
&\left.+\left(g_x(t,y(t),\dot y(t),v(t))^T\lambda(t)\right)_u+\left(h_x(t,y(t),\dot y(t),v(t))^T\mu(t)\right)_u\right\}
\end{aligned}$$

then H_1 is $n\times n$, H_2 is $m\times m$ symmetric matrix, and we can express the above dual (MCD) in the following form

$$\begin{aligned}
\text{(MCD)}\ \max&\int_a^b\{f(t,y(t),\dot y(t),v(t))+\lambda(t)^Tg(t,y(t),\dot y(t),v(t))\\
&+\mu(t)^Th(t,y(t),\dot y(t),v(t))-\frac{1}{2}\beta(t)^T[H_1(t,y(t),\dot y(t),v(t),\lambda(t),\mu(t))]\beta t\\
&-\frac{1}{2}\gamma(t)^T[H_2(t,y(t),\dot y(t),v(t),\lambda(t),\mu(t))]\gamma(t)\}dt
\end{aligned}$$

Subject to

$$\begin{aligned}
&f_x(t,y(t),\dot y(t),v(t))+g_x(t,y(t),\dot y(t),v(t))^T\lambda(t)\\
&+h_x(t,y(t),\dot y(t),v(t))^T\mu(t)-\frac{d}{dt}\left(f_{\dot x}(t,y(t),\dot y(t),v(t))\right.\\
&\left.+g_{\dot x}(t,y(t),\dot y(t),v(t))^T\lambda(x)+h_{\dot x}(t,y(t),\dot y(t),v(t))^T\mu(t)\right)\\
&+[H_1(t,y(t),\dot y(t),v(t),\lambda(t),\mu(t))]\beta(t)=0
\end{aligned}\tag{9}$$

$$\begin{aligned}
&f_u(t,y(t),\dot y(t),v(t))+g_u(t,y(t),\dot y(t),v(t))^T\lambda(t)\\
&+h_u(t,y(t),\dot y(t),v(t))^T\mu(t)+[H_2(t,y(t),\dot y(t),v(t),\lambda(t),\mu(t))]\gamma(t)=0
\end{aligned}\tag{10}$$

$$y(a)=\gamma_1,\ y(b)=\gamma_2 \tag{11}$$

$$\lambda(t)\in\mathbb{R}_+^r,\ \mu(t)\in\mathbb{R}^s,\ \beta(t)\in\mathbb{R}^n,\ \gamma(t)\in\mathbb{R}^m,\ t\in I. \tag{12}$$

In the above problems, $u(t)$ is required to have piecewise continuous first and second derivatives in the interval $a\le t\le b$. $x(t)$ and $\lambda(t)$ are required to be continuous in $a\le t\le b$; $\dot{x}(t)$ and $\mu(t)$ are required to be continuous in $a\le t\le b$ except for values of t corresponding to discontinuities of $u(t)$. The constraints must be fulfilled for all t, $a\le t\le b$, except that for values of t corresponding to points of discontinuity of $u(t)$, (2) and (9) must be fulfilled for right and left hand limits.

Theorem 1. (Weak Duality) Let $(x(t),u(t))$, and $(y(t),v(t),\lambda(t),\mu(t),\beta(t),\gamma(t))$ be feasible solutions of (*CP*) and (*MCD*), respectively. Let $\int_a^b f(t,.,.,.)dt$, $\int_a^b \lambda(t)^T g(t,.,.,.)dt$ and $\int_a^b \mu(t)^T h(t,.,.,.)dt$ be $\rho_0-(\eta,\xi,\theta)-$invex, $\rho_1-(\eta,\xi,\theta)-$invex and $\rho_2-(\eta,\xi,\theta)-$invex functions in x, $\dot{x}$ and $\dot{u}$ on I with respect to the same functions $\eta,\xi,\theta,$ for $\rho_0,\rho_1,\rho_2\in\mathbb{R}$ with $\rho_0+\rho_1+\rho_2\ge 0$. If there exist real valued functions $k(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))>0$, $K(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))>0$ on $I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m$ and $t\in I$ with the following conditions:

$$\beta(t)^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)\ge k(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\,\|\beta(t)\|^2, \tag{13}$$

$$\gamma(t)^T H_2(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\gamma(t)\ge k(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\,\|\gamma(t)\|^2, \tag{14}$$

$$\|H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\le K(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t)), \tag{15}$$

$$\|H_2(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\le K(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t)), \tag{16}$$

$$\frac{1}{2}\|\beta(t)\|\left[k(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\,\|\beta(t)\|-2\|\eta\|\,K(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\right]\ge 0, \tag{17}$$

$$\frac{1}{2}\|\gamma(t)\|\left[k(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\,\|\gamma(t)\|-2\|\xi\|\,K(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\right]\ge 0, \tag{18}$$

then the following inequality holds between the primal (*CP*) and the dual (*MCD*)

$$\begin{aligned}\int_a^b f(t,x(t),\dot{x}(t),u(t))dt \ge \int_a^b \{ & f(t,y(t),\dot{y}(t),v(t))+\lambda(t)^T g(t,y(t),\dot{y}(t),v(t)) \\ & +\mu(t)^T h(t,y(t),\dot{y}(t),v(t))-\frac{1}{2}\beta(t)^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t) \\ & -\frac{1}{2}\gamma(t)^T H_2(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\gamma(t)\}\,dt\end{aligned}$$

Proof: Since $(x(t),u(t))$, and $(y(t),v(t),\lambda(t),\mu(t),\beta(t),\gamma(t))$ are feasible solutions of (*CP*) and (*MCD*), respectively, we obtain

$$\int_a^b f(t,x(t),\dot{x}(t),u(t))dt - \int_a^b \big[f(t,y(t),\dot{y}(t),v(t)) + \lambda(t)^T g(t,y(t),\dot{y}(t),v(t))$$

$$+\mu(t)^T h(t,y(t),\dot{y}(t),v(t)) - \frac{1}{2}\beta(t)^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)$$

$$-\frac{1}{2}\gamma(t)^T H_2(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\gamma(t)\big\} dt$$

$$\geq \left[\int_a^b f(t,x(t),\dot{x}(t),u(t))dt - \int_a^b f(t,y(t),\dot{y}(t),v(t))dt \right]$$

$$+\left[\int_a^b \lambda(t)^T g(t,x(t),\dot{x}(t),u(t))dt - \int_a^b \lambda(t)^T g(t,y(t),\dot{y}(t),v(t))dt \right]$$

$$+\left[\int_a^b \mu(t)^T h(t,x(t),\dot{x}(t),u(t))dt - \int_a^b \mu(t)^T h(t,y(t),\dot{y}(t),v(t))dt \right]$$

$$+\frac{1}{2}\int_a^b \beta(t)^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)dt$$

$$+\frac{1}{2}\int_a^b \gamma(t)^T H_2(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\gamma(t)dt$$

(by (1), (2) and (32))

$$\geq \int_a^b \eta^T \big[f_x(t,y(t),\dot{y}(t),v(t)) + g_x(t,y(t),\dot{y}(t),v(t))^T \lambda(t)$$

$$+h_x(t,y(t),\dot{y}(t),v(t))^T \mu(t) - \frac{d}{dt}\big(f_{\dot{x}}(t,y(t),\dot{y}(t),v(t))$$

$$+g_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T \lambda(x) + h_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T \mu(t)\big)\big] dt$$

$$+\int_a^b \xi^T \big[f_u(t,y(t),\dot{y}(t),v(t)) + g_u(t,y(t),\dot{y}(t),v(t))^T \lambda(t)$$

$$+h_u(t,y(t),\dot{y}(t),v(t))^T \mu(t) + \frac{1}{2}\beta(t)^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)\big] dt$$

$$+\frac{1}{2}\int_a^b \gamma(t)^T H_2(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\gamma(t)dt$$

$$+\int_a^b (\rho_0 + \rho_1 + \rho_2) \| \theta(t,x(t),\dot{x}(t),y(t),\dot{y}(t),u(t),v(t) \|^2 \, dt$$

(by integrating by parts and $\eta = 0$ at $t = a$ and $t = b$, and invexity of $\int_a^b f(t,.,.,.)dt$, $\int_a^b \lambda(t)^T (t,.,.,.)dt$, and $\int_a^b \mu(t)^T f(t,.,.,.)dt$)

$$\geq \int_a^b \big[-\eta^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t) - \xi^T H_2(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\gamma(t) \big] dt$$

$$+\int_a^b \frac{1}{2}\big[\beta(t)^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t) + \gamma(t)^T H_2(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\gamma(t) \big] dt$$

(by (9) and (10); and $\rho_0 + \rho_1 + \rho_2 \geq 0$)

$$\geq \int_a^b [-\|\eta\| K(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\beta(t)\| + \frac{1}{2}k(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\beta(t)\|^2]\,dt$$

$$+\int_a^b [-\|\xi\| K(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\gamma(t)\| + \frac{1}{2}k(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\gamma(t)\|^2]\,dt$$

(by Schwarz inequality, and (13), (14), (15), (16))

$\geq 0.$

(by (17), (18))

Hence the result.

We construct the following example which verifies the above Theorem 1, in which both the objective and constraint functions are $\rho-(\eta,\xi,\theta)-$ invex. The following example is similar to that of Example 1 in [3].

Example 4. Let us define $f, g : I\times[0,1]\times[0,1]\times[0,1]\to\mathbb{R}$ by

$$f(t,x(t),\dot{x}(t),u(t))] = \frac{-x^2(t)u^2(t)t}{b^2-a^2},\ t\in I=[a,b],\ a,b\geq 0$$

$$\lambda(t)^T g(t,x(t),\dot{x}(t),u(t)) = \frac{-\lambda(t)(x^2(t)u^2(t)+1)t}{b^2-a^2},\ t\in I=[a,b],\ a,b\geq 0.$$

$$\mu(t)^T h(t,x(t),\dot{x}(t),u(t)) = \frac{-\mu(t)(x^2(t)u^2(t)-\frac{1}{12}x(t)u(t))t}{b^2-a^2},\ t\in I=[a,b],\ a,b\geq 0.$$

$\int_a^b f(t,.,.,.)\,dt$ is $\frac{1}{12}-(\eta,\xi,\theta)-$ invex, $\int_a^b \lambda(t)^T g(t,.,.,.)\,dt$ is $\frac{1}{12}-(\eta,\xi,\theta)-$ invex, and $\int_a^b \mu(t)^T h(t,.,.,.)\,dt$ is $-\frac{1}{6}-(\eta,\xi,\theta)-$ invex, with respect to same $\eta : I\times[0,1]^6\to\mathbb{R}$, $\xi : I\times[0,1]^6\to\mathbb{R}$ and $\theta : I\times[0,1]^6\to\mathbb{R}$ defined as

$$\eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t)) = \begin{cases} y(t)(y(t)v(t)-x(t)u(t), & \text{if } y(t)v(t) > x(t)u(t) \\ 0, & \text{if } y(t)v(t)\leq x(t)u(t)\end{cases}$$

$$\xi(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t)) = \begin{cases} -v(t)(y(t)v(t)-x(t)u(t), & \text{if } y(t)v(t) > x(t)u(t) \\ 0, & \text{if } y(t)v(t)\leq x(t)u(t)\end{cases}$$

$$\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t)) = \begin{cases} \sqrt{y(t)v(t)-x(t)u(t)}, & \text{if } y(t)v(t) > x(t)u(t) \\ 0, & \text{if } y(t)v(t)\leq x(t)u(t)\end{cases}$$

Here it is easily verified that $\int_a^b f(t,.,.,.)\,dt$, $\int_a^b \lambda(t)^T g(t,.,.,.)\,dt$ and $\int_a^b \mu(t)^T h(t,.,.,.)\,dt$ are $\frac{1}{12}-(\eta,\xi,\theta)-$ invex, $\frac{1}{12}-(\eta,\xi,\theta)-$ invex and $-\frac{1}{6}-(\eta,\xi,\theta)-$ invex functions respectively.

Now by taking $x(t)=\frac{1}{3}$, $u(t)=\frac{1}{4}$, $y(t)=t(1-t)$ and $v(t) = 1$ for all $t\in[0,1]$. Here $a = 0$, $b = 1$, and taking $t=\frac{1}{3}$ the primal feasible and the dual feasible solutions are –0.0040 and –0.2981.

Necessary conditions for the existence of an extremal solution for a variational problem subject to the both equality and inequality constraints were given by Valentine [16]. Using Valentine's results, Berkovitz [2] obtained the corresponding necessary conditions for the control problem (CP). These may be stated in the following way. If ($y(t)$, $v(t)$) is an optimal solution for (CP), then

$$\mu_0 f_x(t,y(t),\dot{y}(t),v(t))+g_x(t,y(t),\dot{y}(t),v(t))^T\lambda(t)+h_x(t,y(t),\dot{y}(t),v(t))^T\mu(t)$$
$$=\frac{d}{dt}\left(\mu_0 f_{\dot{x}}(t,y(t),\dot{y}(t),v(t))+g_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T\lambda(t)+h_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T\mu(t)\right) \tag{19}$$

$$\mu_0 f_u(t,y(t),\dot{y}(t),v(t))+g_u(t,y(t),\dot{y}(t),v(t))^T\lambda(t)+h_u(t,y(t),\dot{y}(t),v(t))^T\mu(t)=0 \tag{20}$$

$$\lambda(t)^T g_u(t,y(t),\dot{y}(t),v(t))=0 \tag{21}$$

$$\lambda(t)\in\mathbb{R}^r_+ \tag{22}$$

hold throughout $a\le t\le b$ (except for the values of t corresponding to points of discontinuity of $u(t)$, (19) holds for right and left hand limits). Here μ_0 is nonnegative constant, $\mu(t)$ is continuous in a $a\le t\le b$, and μ_0, $\lambda(t)$, $\mu(t)$ can not vanish simultaneously for any $a\le t\le b$. It will be assumed that the minimizing arc determined by $y(t)$, $v(t)$ is normal, i.e., that μ_0 can be taken equal to 1.

Theorem 2. (Strong Duality) Let the Weak Duality Theorem 2.1holds between the control primal (*CP*) and the Mangasarian dual (*MCD*). If ($y(t)$, $v(t)$) be an optimal solution of (*CP*), then there exist functions $\lambda(t)$ and $\mu(t)$ such that $(y(t),v(t),\lambda(t),\mu(t),\beta(t)=0,\gamma(t)=0)$ is an optimal solution of (*MCD*) and the optimal values of (*CP*) and (*MCD*) are equal.

Proof: It follows from Berkovitz results [2] that there exist $\lambda(t)$ and $\mu(t)$ such that $(y(t),v(t),\lambda(t),\mu(t),\beta(t)=0,\gamma(t)=0)$ satisfies the constraints of (*MCD*). Equations (2), (21) and Theorem 1 shows that optimal values of (*CP*) and (*MCD*) are equal.

Theorem 3. (Converse Duality) Let $(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t),\beta(t),\gamma(t))$ be an optimal solution of (*MCD*). Suppose that $\int_a^b f(t,.,.,.)\,dt$ is $\rho_0-(\eta,\xi,\theta)-$invex, $\int_a^b \lambda(t)^T g(t,.,.,.)\,dt$ is $\rho_1-(\eta,\xi,\theta)-$invex and $\int_a^b \lambda(t)^T g(t,.,.,.)\,dt$ is $\rho_2-(\eta,\xi,\theta)-$invex functions with respect to the same η,ξ,θ, and $\rho_0+\rho_1+\rho_2\ge 0$. Also assume that

$$\int_a^b\left[\lambda(t)^T g(t,y(t),\dot{y}(t),v(t))-\eta^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)\right]dt\ge 0. \tag{23}$$

$$\int_a^b\left[\mu(t)^T h(t,y(t),\dot{y}(t),v(t))-\xi^T H_2(t,y(t),\dot{y}(t),v(t),\mu(t),\mu(t))\gamma(t)\right]dt\ge 0. \tag{24}$$

Then ($y(t)$, $v(t)$) is an optimal solution of (*CP*).

Proof: Suppose that ($y(t)$, $v(t)$) is not an optimal solution of (*CP*). Then there exists a feasible solution $(\bar{x}(t),\bar{u}(t))$ of the primal (*CP*) such that

$$\int_a^b f(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t))\,dt<\int_a^b f(t,y(t),\dot{y}(t),v(t))\,dt. \tag{25}$$

Given that $\int_a^b f(t,.,.,.)\,dt$ is $\rho_0-(\eta,\xi,\theta)-$ invex, $\int_a^b \lambda(t)^T g(t,.,.,.)\,dt$ is $\rho_1-(\eta,\xi,\theta)-$ invex and $\int_a^b \mu(t)^T h(t,.,.,.)\,dt$ is $\rho_2-(\eta,\xi,\theta)-$ invex with respect to the same η,ξ and θ. We have

$$\int_a^b f(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t))\,dt-\int_a^b f(t,y(t),\dot{y}(t),v(t))\,dt \geq \int_a^b \Big\{\eta^T f_{\bar{x}}(t,y(t),\dot{y}(t),v(t))$$
$$+\left(\frac{d}{dt}\eta^T\right) f_{\dot{\bar{x}}}(t,y(t),\dot{y}(t),v(t))+\xi^T f_{\bar{u}}(t,y(t),\dot{y}(t),v(t)) \tag{26}$$
$$+\rho_0\,\|\theta(t,x(t),\dot{x}(t),y(t),\dot{y}(t),u(t),v(t))\|^2\Big\}\,dt$$

using eqn. (9) in eqn. (26), we obtain

$$\int_a^b f(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t))\,dt-\int_a^b f(t,y(t),\dot{y}(t),v(t))\,dt$$
$$\geq \int_a^b \eta^T\Big[-g_{\bar{x}}(t,y(t),\dot{y}(t),v(t))^T\lambda(t)-h_{\bar{x}}(t,y(t),\dot{y}(t),v(t))^T\mu(t)$$
$$+\frac{d}{dt}\big\{g_{\dot{\bar{x}}}(t,y(t),\dot{y}(t),v(t))^T\lambda(t)+h_{\dot{\bar{x}}}(t,y(t),\dot{y}(t),v(t))^T\mu(t)\big\}$$
$$-H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)\Big]+\xi^T f_{\bar{u}}(t,y(t),\dot{y}(t),v(t))$$
$$+\rho_0\,\|\theta(t,x(t),\dot{x}(t),y(t),\dot{y}(t),u(t),v(t))\|^2\Big\}\,dt$$

$$=\int_a^b -\eta^T\Big[g_{\bar{x}}(t,y(t),\dot{y}(t),v(t))^T\lambda(t)-\frac{d}{dt}g_{\dot{\bar{x}}}(t,y(t),\dot{y}(t),v(t))^T\lambda(t)\Big]$$
$$+\int_a^b -\eta^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)dt+\int_a^b \xi^T f_{\bar{u}}(t,y(t),\dot{y}(t),v(t))\,dt$$
$$+\int_a^b \rho_0\,\|\theta(t,x(t),\dot{x}(t),y(t),\dot{y}(t),u(t),v(t))\|^2\,dt$$

$$\geq \int_a^b \Big[\lambda(t)^T g(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t))+\lambda(t)^T g(t,y(t),y'(t),v(t))+\xi(t)^T g_{\bar{u}}(t,y(t),\dot{y}(t),v(t))^T\lambda(t)$$
$$+\rho_1\,\|\theta(t,x(t),\dot{x}(t),y(t),\dot{y}(t),u(t),v(t))\|^2\Big\}dt+\int_a^b\Big[-\mu(t)^T h(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t))$$

$$+\mu(t)^T h(t,y(t),y'(t),v(t))+\xi(t)^T h_{\bar{u}}(t,y(t),\dot{y}(t),v(t))^T\mu(t)$$
$$+\rho_2\,\|\theta(t,x(t),\dot{x}(t),y(t),\dot{y}(t),u(t),v(t))\|^2\Big\}dt$$
$$+\int_a^b -\eta^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)dt+\int_a^b \xi^T f_{\bar{u}}(t,y(t),\dot{y}(t),v(t))\,dt$$
$$+\int_a^b \rho_0\,\|\theta(t,x(t),\dot{x}(t),y(t),\dot{y}(t),u(t),v(t))\|^2\,dt$$

(by invexity of $\int_a^b \lambda(t)^T g(t,.,.,.)\,dt$ and $\int_a^b \mu(t)^T h(t,.,.,.)\,dt$)
≥ 0.
(by (39), (40) and $\rho_0+\rho_1+\rho_2\geq 0$)
which is a contradiction. Hence the result follows.

Mond-Wier Type Duality

Mond-Wier second order dual (*MWCD*) of the control primal problem (*CP*) is

$$\text{(MWCD)} \max \int_a^b f(t, y(t), \dot{y}(t), v(t))dt$$

Subject to

$$\begin{aligned} & f_x(t, y(t), \dot{y}(t), v(t)) + g_x(t, y(t), \dot{y}(t), v(t))^T \lambda(t) \\ & + h_x(t, y(t), \dot{y}(t), v(t))^T \mu(t) - \frac{d}{dt}\big(f_{\dot{x}}(t, y(t), \dot{y}(t), v(t)) \\ & + g_{\dot{x}}(t, y(t), \dot{y}(t), v(t))^T \lambda(t) + h_{\dot{x}}(t, y(t), \dot{y}(t), v(t))^T \mu(t)\big) \\ & + [H_1(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t))]\beta(t) = 0 \end{aligned} \tag{27}$$

$$\begin{aligned} & f_u(t, y(t), \dot{y}(t), v(t)) + g_u(t, y(t), \dot{y}(t), v(t))^T \lambda(t) \\ & + h_u(t, y(t), \dot{y}(t), v(t))^T \mu(t) + [H_2(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t))]\gamma(t) = 0 \end{aligned} \tag{28}$$

$$\lambda(t)^T g(t, y(t), \dot{y}(t), v(t)) - \beta(t)^T [H_1(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t))]\beta(t) = 0 \tag{29}$$

$$\mu(t)^T h(t, y(t), \dot{y}(t), v(t)) - \gamma(t)^T [H_1(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t))]\gamma(t) = 0 \tag{30}$$

$$y(a) = \gamma_1,\ y(b) = \gamma_2 \tag{31}$$

$$\lambda(t) \in \mathbb{R}_+^r,\ \mu(t) \in \mathbb{R}^s,\ \beta(t) \in \mathbb{R}^n,\ \gamma(t) \in \mathbb{R}^m,\ t \in I. \tag{32}$$

Theorem 4. (Weak Duality) Let ($x(t)$, $u(t)$), and $(y(t), v(t), \lambda(t), \mu(t), \beta(t), \gamma(t))$ be feasible solutions of (*CP*) and (*MWCD*), respectively. Let $\int_a^b f(t,.,.,.)\,dt$, $\int_a^b \lambda(t)^T g(t,.,.,.)\,dt$ and $\int_a^b \mu(t)^T h$ $(t,.,.,.)\,dt$ be $\rho_0-(\eta,\xi,\theta)-$invex, $\rho_1-(\eta,\xi,\theta)-$invex and $\rho_2-(\eta,\xi,\theta)-$invex functions in x, $\dot{x}$ and $\dot{u}$ on I with respect to the same functions η, ξ, θ for ρ_0, ρ_1, $\rho_2 \geq \mathbb{R}$ with $\rho_0+\rho_1+\rho_2 \geq 0$. If there exist real valued functions $k(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) > 0$, $K(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) > 0$ on $I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ and $t \in I$ with the following conditions:

$$\beta(t)^T H_1(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t))\beta(t) \geq k(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) \| \beta(t) \|^2, \tag{33}$$

$$\gamma(t)^T H_2(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t))\gamma(t) \geq k(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) \| \gamma(t) \|^2, \tag{34}$$

$$\| H_1(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) \| \leq K(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)), \tag{35}$$

$$\| H_2(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) \| \leq K(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)), \tag{36}$$

$$\| \beta(t) \| \left[k(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) \| \beta(t) \| - \| \eta \| K(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t))\right] \geq 0, \tag{37}$$

$$\|\gamma(t)\|\left[k(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\gamma(t)\|-\|\xi\|K(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\right]\ge 0, \tag{38}$$

then the following inequality holds between the primal (*CP*) and the dual (*MWCD*)

$$\int_a^b f(t,x(t),\dot{x}(t),u(t))dt \ge \int_a^b f(t,y(t),\dot{y}(t),v(t))dt$$

Proof: Since ($x(t)$, $u(t)$), and $(y(t),v(t),\lambda(t),\mu(t),\beta(t),\gamma(t))$ are feasible solutions of (*CP*) and (*MWCD*), respectively, we obtain

$$\int_a^b f(t,x(t),\dot{x}(t),u(t))dt-\int_a^b f(t,y(t),\dot{y}(t),v(t))dt$$
$$\ge \int_a^b\Big[\eta^T f_x(t,y(t),\dot{y}(t),v(t))+\left(\frac{d}{dt}\eta^T\right)f_{\dot{x}}(t,y(t),\dot{y}(t),v(t))$$
$$+\xi^2 f_u(t,y(t),\dot{y}(t),v(t))+\rho_0\|\theta(t,x(t),\dot{x}(t),y(t),\dot{y}(t),u(t),v(t))\|^2\Big]dt$$

(by integrating by parts and $\eta=0$ at t = a and $t = b$, and invexity of $\int_a^b f(t,.,.,.)\,dt$)

$$=\int_a^b\Big\{\eta^T\Big[-g_x(t,y(t),\dot{y}(t),v(t))^T\lambda(t)-h_x(t,y(t),\dot{y}(t),v(t))^T\mu(t)$$
$$+\frac{d}{dt}\left(g_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T\lambda(t)+h_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T\mu(t)\right)$$
$$-H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)\Big]+\xi^T f_u(t,y(t),\dot{y}(t),v(t))$$
$$+\rho_0\|\theta(t,x(t),\dot{x}(t),y(t),\dot{y}(t),u(t),v(t))\|^2\Big\}dt$$

(by eqn. (27))

$$\ge\int_a^b\lambda(t)^T g(t,y(t),\dot{y}(t),v(t))\,dt+\int_a^b\mu(t)^T h(t,y(t),\dot{y}(t),v(t))\,dt$$
$$+\int_a^b\Big\{\xi^T\big[f_u(t,y(t),\dot{y}(t),v(t))+g_u(t,y(t),\dot{y}(t),v(t))^T\lambda(t)$$
$$h_u(t,y(t),\dot{y}(t),v(t))^T\mu(t)\big]-\eta^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)\Big\}dt$$

(by invexity of $\int_a^b\lambda(t)^T g(t,.,.,.)\,dt$, $\int_a^b\mu(t)^T h(t,.,.,.)\,dt$, (1), (2), (8)
and $\rho_0+\rho_1+\rho_2\ge 0$

$$=\int_a^b\Big[\beta(t)^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)+\gamma(t)^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\gamma(t)$$
$$-\xi^T H_2(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\gamma(t)-\eta^T H_1(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\beta(t)\Big]dt$$

$$\ge\int_a^b\Big[k(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\beta(t)\|^2+k(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\gamma(t)\|^2$$
$$-\|\xi^T\|K(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\gamma(t)\|-\|\eta\|K(t,y(t),\dot{y}(t),v(t),\lambda(t),\mu(t))\|\beta(t)\|\Big]dt$$

(by Schwarz inequality)

$$= \int_a^b \| \beta(t) \| \left[k(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) \| \beta(t) \| - \| \eta \| K(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) \right] dt$$
$$+ \int_a^b \| \gamma(t) \| \left[k(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) \| \gamma(t) \| - \| \xi \| K(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) \right] dt$$

$\geq 0.$

(by (37), (38))

Hence the result.

Theorem 5. (Strong Duality) Let the Weak Duality Theorem 4 holds between the control primal (*CP*) and the Mond-Weir dual (MWCD). If ($y(t)$, $v(t)$) be an optimal solution of (*CP*), then there exist functions $\lambda(t)$ and $\mu(t)$ such that $(y(t), v(t), \lambda(t), \mu(t), \beta(t) = 0, \gamma(t) = 0)$ is an optimal solution of (MWCD) and the optimal values of (*CP*) and (MWCD) are equal.

Proof: The proof is similar to that of Theorem 2.

Theorem 6. (Converse Duality)

Let $(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t), \beta(t), \gamma(t))$ be an optimal solution of (MWCD). Suppose that $\int_a^b f(t,.,.,.)\,dt$ is $\rho_0 - (\eta, \xi, \theta) -$ invex, $\int_a^b \lambda(t)^T g(t,.,.,.)\,dt$ is $\rho_1 - (\eta, \xi, \theta) -$ invex and $\int_a^b \lambda(t)^T g(t,.,.,.)\,dt$ is $\rho_2 - (\eta, \xi, \theta) -$ invex functions with respect to the same η, ξ, θ, and $\rho_0 + \rho_1 + \rho_2 \geq 0$. Also assume that

$$\int_a^b \left[\lambda(t)^T g(t, y(t), \dot{y}(t), v(t)) - \eta^T H_1(t, y(t), \dot{y}(t), v(t), \lambda(t), \mu(t)) \beta(t) \right] dt \geq 0. \tag{39}$$

$$\int_a^b \left[\mu(t)^T h(t, y(t), \dot{y}(t), v(t)) - \xi^T H_2(t, y(t), \dot{y}(t), v(t), \mu(t), \mu(t)) \gamma(t) \right] dt \geq 0. \tag{40}$$

Then ($y(t)$, $v(t)$) is an optimal solution of (*CP*).

Proof: The proof is similar to that of Theorem 3.

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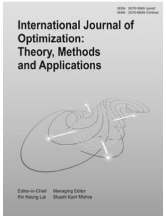

Indefinite Quadratic Bilevel Programming Problem with Multiple Objectives at Both Levels

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Abstract. Bilevel Programming has been proposed for dealing with decision process involving two decision makers with a hierarchical structure. They are characterized by the existence of two optimization problems in which the constraint region of the upper level problem is implicitly determined by the lower level optimization problem. In this paper, a general bilevel optimization problem with multiple objectives at both levels is considered. The objective functions at both levels are indefinite quadratic and the feasible region is assumed to be a convex polyhedron. An algorithm is developed to find an efficient solution of the bilevel programming problem.

Keywords: indefinite quadratic programming problem, Multi-objective programming, bilevel programming, efficient solution, quasi- concave function.

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Introduction

The bilevel programming structure is a class of hierarchical problem that show a two-stage decision making process when the constraint region of the first level problem is implicitly determined by another optimization problem.

General Bilevel Programming Problem (BLPP)is defined as,

(BLPP): $\underset{X}{\text{Max}}\, f(X,Y)$

where Y solves

$\underset{Y}{\text{Max}}\, F(X,Y)$

subject to $(X, Y) \in S$,

where $S = \{(X,Y) : AX + BY \le b;\ X,Y \ge 0\}$.

Here, $f(X, Y)$ and $F(X, Y)$ can be linear or non-linear. (BLPP) has been used by researchers in several fields ranging from economics to transportation engineering. (BLPP) is also used to model problems involving multiple decision makers. These problems include traffic signal optimization [12] and genetic algorithms [4]. There are numerous methods to solve (BLPP). The most notable among them are cutting plane method [1,8,9], branch and bound methods [6,11,13] and the ranking method [2,10].

There are many planning and /or decision making situations that can be properly represented by a multi-objective programming model. Mathematically, a multi-objective programming problem (MOPP), is defined as,

(MOPP): $\text{Max}\{f_1(X) = Z_1\}$

$\text{Max}\{f_2(X) = Z_2\}$

$\cdots$

$\text{Max}\{f_k(X) = Z_k\}$

subject to $X \in S$,

where S is a feasible set and $f_1(X); \{j = 1, \cdots, k\}$, be linear/non-linear.

In the late seventies and early eighties, a lot of papers have been published on (MOPP) problems. White [14], in 1996, presented several equivalent mathematical programming formulation of the problem for maximizing a function over the efficient set, in case of a polytopal feasible region and linear function.

Indefinite Quadratic Programming Problem

Consider the indefinite quadratic programming problem (IQPP), given by

$$\text{(IQPP)}: \quad \text{Max } Z(X) = Z_1(X)Z_2(X) = (C^T X + \alpha)(D^T X + \beta)$$
$$\text{Subject to } AX \le b$$
$$X \ge 0,$$

where $X \in \mathbb{R}^n; b \in \mathbb{R}^m; C, D \in \mathbb{R}^n; \alpha, \beta \in \mathbb{R}$ and $A \in \mathbb{R}^{mxn}$.

The feasible region $S = \{X : AX \le b; X \ge 0\}$ is non-empty and bounded.

Both $Z_1(X)$ and $Z_2(X)$ are positive for all $X \in S$.

Thus, the function $Z(X)$ is both quasi-concave and quasi-convex on S. Therefore, the optimal solution to the problem (IQPP) occurs at an extreme point of S.

Indefinite Quadratic Bilevel Programming Problem with Multiple Objective Functions at the Upper Level

The indefinite quadratic multi-objective bilevel programming problem (IQMBPP) can be formulated as,

$$\text{(IQMBPP)}: \quad \underset{X}{\text{Max}}(f_1(X,Y), f_2(X,Y), \cdots, f_k(X,Y))$$
$$\text{subject to } A_1^1 X + A_2^1 Y \le b^1$$
$$X, Y \ge 0$$
$$\text{where for a given } X, Y \text{ solves}$$
$$\underset{Y}{\text{Max}}\, g(X,Y)$$
$$\text{subject to } A_1^2 X + A_2^2 Y \le b^2$$
$$Y \ge 0,$$

where $f_i(X,Y) = Z_{i1}(X,Y).\, Z_{i2}(X,Y); \quad i = 1, \cdots, k$

$$Z_{i1}(X,Y) = c_{i1}X + d_{i1}Y + \alpha_i; \quad i = 1, \cdots, k$$
$$Z_{i2}(X,Y) = c_{i2}X + d_{i2}Y + \beta_i; \quad i = 1, \cdots, k$$
$$g(X,Y) = (p_1 X + p_2 Y + \gamma)\,(q_1 X + q_2 Y + \delta).$$

Here, $X = \{x_1, \cdots, x_{n_1}\} \in \mathbb{R}^{n_1}$, $Y = \{y_1, \cdots, y_{n_1}\} \in \mathbb{R}^{n_2}$

$$c_{i1}, c_{i2} \in \mathbb{R}^{n_1}; \quad i = 1, \cdots, k; \quad d_{i1}, d_{i2} \in \mathbb{R}^{n_2}; \quad i = 1, \cdots, k$$
$$\alpha_i, \beta_i \in \mathbb{R}; \quad i = 1, \cdots, k.$$
$$A_1^1 \in \mathbb{R}^{m_1 \times n_1}; \quad A_2^1 \in \mathbb{R}^{m_1 \times n_2}; \quad b^1 \in \mathbb{R}^{m_1},$$
$$A_1^2 \in \mathbb{R}^{m_2 \times n_1}; \quad A_2^2 \in \mathbb{R}^{m_2 \times n_2}; \quad b^2 \in \mathbb{R}^{m_2},$$
$$p_1, p_2 \in \mathbb{R}^{n_1}; \quad q_1, q_2 \in \mathbb{R}^{n_2}; \quad \gamma, \delta \in \mathbb{R}$$

The objective functions at both the levels are the product of two positive valued affine functions, hence, they are quasi-concave.

Let $S_0 = \{(X, Y): A_1^1 X + A_2^1 Y \le b^1;\ A_1^2 X + A_2^2 Y \le b^2;\ X, Y \ge 0\}$.

The polyhedron S_0 defined by the constraint region of the (IQMBPP) problem is assumed to be non-empty and compact.

Define the lower level problem as follows:

For a given $X = \bar{X} \ge 0$, Y solves

$$\text{(IQMBPP}(\bar{X})): \quad \max_{Y} g(\bar{X}, Y) = (p_1\bar{X} + p_2 Y + \gamma)\,(q_1\bar{X} + q_2 Y + \delta)$$
$$\text{Subject to } A_2^2 Y \le b^2 - A_1^2 \bar{X}$$
$$Y \ge 0,$$

The feasible region of the lower level problem is $S(X) = \{Y : A_2^2 Y \le b^2 - A_1^2 X;\ Y \ge 0\}$.

The inducible region or the feasible region of the upper level problem is defined as $\bar{S} = \{(X, Y) \in S_0 : Y \text{ solves } \max_Y g(X, Y), \text{ for a given } X\}$.

It is assumed that $\bar{S}$ is non-empty and the optimal solution of (IQMBPP ($\bar{X}$)) is singleton.

Bilevel Feasible Solution

Since $\bar{S}$ is non-empty, therefore, for each value of the upper level problem, the lower problem has a feasible solution. Thus, any point of $\bar{S}$ is a bilevel feasible solution.

Efficient Solution

A bilevel feasible solution $(\bar{X}, \bar{Y}) \in \bar{S}$ is an efficient solution of (IQMBPP) if there is no $(X, Y) \in \bar{S}$, such that $f_{i(\bar{X},\bar{Y})} \le f_i(X, Y)$, for $i = 1, \cdots, k$ and $f_{j(\bar{X},\bar{Y})} < f_j(X, Y)$, for some $j \in \{1, \cdots, k\}$.

Efficient Set

The set of all efficient solutions is denoted by (SE) and is called the efficient set.

Solving (IQMBPP) Problem

In order to solve (IQMBPP) problem, consider the multi-objective problem at the upper level, given by

$$\text{(IQMULPP):} \quad \max_{X} (f_1(X, Y), f_2(X, Y), \cdots, f_k(X, Y))$$
$$\text{subject to } (X, Y) \in S_0.$$

Since S_0 is non-empty and compact and $f_i's (i = 1, \cdots, k)$ are quasi-concave function, therefore, the optimal solution corresponding to each objective function $f_i(X, Y), i = 1, \cdots, k,$ lies at an extreme point of S_0. If this is also a point of $\bar{S}$, then it is an extreme point of inducible region $\bar{S}$, [3].

Consider (IQMULPP) problem with the first objective function $f_1(X, Y)$. Let its optimal solution be (X^1, Y^1). Because of our assumption that optimal solution is unique, it becomes an efficient solution of the given problem.

Otherwise, put $X = X^1$ in the lower level problem. Let the optimal solution of this problem be (X^1, Y_1).

If $Y^1 = Y_1$, then (X^1, Y^1) is a bilevel feasible solution. Find that (X^1, Y^1) which is an efficient solution of (IQMBPP) problem.

If $Y^1 \neq Y_1$, then solve the problem (IQMULPP) with second objective function and repeat the process till an efficient solution of (IQMBPP) problem is obtained. If not, then find the second best solution of first objective function and repeat the process with the objective functions $f_i(X,Y)$; $i = 2,\cdots,k.$ Finally, we will get an extreme point which is an efficient solution of (IQMBPP) problem because extreme points of $\overline{S}$ are contained in S_0.

Algorithm to Solve (IQMBPP) Problem

Step 1 Consider the (IQMULPP) problem as

$$\underset{X}{\text{Max}}((f_1(X,Y),\cdots,f_k(X,Y))$$

subject to $(X, Y) \in S_0$

Step 2 Set $i = 1$.

Let (X^i, Y^i) be an optimal solution of $f_i(X, Y)$ for $i = 1,\cdots,k$.

Step 3 Put $X = X^i$ in the follower's problem, as

$$\underset{Y}{\text{Max}}\; g(X^i,Y) = (p_1X^i + p_2Y + \gamma)(q_1X^i + q_2Y + \delta)$$

subject to $A_2^2 Y \le b^2 - A_1^2 X^i$

$Y \ge 0.$

Let (X^i, y) be its optimal solution.

Step 4.1 If $Y^i = y$, then (X^i, Y^i) is a bilevel feasible solution of (IQMBPP)

4.2 If $Y^i \neq y$, set $i = i + 1$.
Go to step 2 and repeat the process.

4.3 If no bilevel feasible solution is obtained, find the next best solution of the first objective function. Go to step 3 and repeat the process with $f_i(X, Y)$; $i = 2,\ldots,k$.

Step 5 From the set of bilevel feasible solutions, find the efficient solutions of the problem (IQMBPP).

Example 1: Consider the following indefinite quadratic bilevel programming problem with multi-objective functions at the upper level:

(IQMBPP): $\underset{x_1}{\text{Max}}\; f_1(x_1,x_2,x_3) = (x_1 + 2x_3 + 3)(3x_2 + 2)$

$$\underset{x_1}{\text{Max}}\ f_2(x_1,x_2,x_3)=(2x_1+x_2+2)(x_3+1)$$
$$\text{subject to } 3x_1+x_2+2x_3\le 5$$
$$x_2+x_3\le 3$$
$$\text{where } (x_2,x_3) \text{ solves}$$
$$\text{Max } g(x_1,\ x_2\ ,x_3)=(x_2+1)(x_1+x_2+x_3+3)$$
$$\text{subject to } x_1+2x_2+x_3\le 2$$
$$3x_2+2x_3\le 6$$
$$x_1,x_2,x_3\ge 0.$$

Solution: Consider the multi-objective problem at the upper level, given by

$$\text{(IQMULPP):}\quad \underset{x_1}{\text{Max}}\ f_1(x_1,x_2,x_3)=(x_1+2x_3+3)(3x_2+2)$$
$$\underset{x_1}{\text{Max}}\ f_2(x_1,x_2,x_3)=(2x_1+x_2+2)(x_3+1)$$
$$\text{subject to } 3x_1+x_2+2x_3\le 5$$
$$x_2+x_3\le 3$$
$$x_1+2x_2+x_3\le 2 \qquad (1)$$
$$3x_2+2x_3\le 6$$
$$x_1,x_2,x_3\ge 0.$$

Solve the first objective

$$\underset{x_1}{\text{Max}}\ f_1(x_1,x_2,x_3)=(x_1+2x_3+3)(3x_2+2)$$

subject to the constraints (1).
The optimal table for the above problem is

			$c_j \rightarrow$	1	0	2	0	0	0	0
			$d_j \rightarrow$	0	3	0	0	0	0	0
C_B	D_B	V_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	x_4	4	$5/2$	0	$3/2$	1	0	$-1/2$	0
0	0	x_5	2	$-1/2$	0	$1/2$	0	1	$-1/2$	0
0	3	x_2	1	$1/2$	1	$1/2$	0	0	$1/2$	0
0	0	x_7	3	$-3/2$	0	$1/2$	0	0	$-3/2$	1
$Z_{11}=$	3	$z_j^{11}-$	$c_j \rightarrow$	-1	0	-2	0	0	0	0
$Z_{12}=$	5	$z_j^{12}-$	$d_j \rightarrow$	$3/2$	0	$3/2$	0	0	$3/2$	0
		L_j	$\rightarrow$	$\frac{-1}{2}+\frac{3}{2}\theta$	0	$\frac{-11}{2}+3\theta$	0	0	$\frac{9}{2}$	0

The optimal solution of $f_1(x_1, x_2, x_3)$ is (0, 1, 0).
Put $x_1 = 0$ in the lower level problem, we get

$$\text{Max } g(0,x_2,x_3)=(x_2+1)(x_2+x_3+3)$$

$$\begin{aligned} \text{subject to } & 2x_2 + x_3 \le 2 \\ & 3x_2 + 2x_3 \le 6 \\ & x_2, x_3 \ge 0. \end{aligned}$$

The optimal solution of the lower level problem is (1, 0).

Thus, (0, 1, 0) is a bilevel feasible solution.

Again, repeating the process with the second objective function, $f_2(x_1, x_2, x_3)$, of the upper level, we get $x_1 = 1$, $x_2 = 0$, $x_3 = 1$.

Putting $x_1 = 1$ in the lower level problem, we find that (1, 0, 1) is not a bilevel feasible solution.

Finding the second best solution of $f_1(x_1, x_2, x_3)$, we find that $\left(\frac{8}{5}, \frac{1}{5}, 0\right)$ is a bilevel feasible solution. Repeating the same method for $f_2(x_1, x_2, x_3)$, bilevel feasible solution is $\left(\frac{8}{5}, \frac{1}{5}, 0\right)$.

Thus, the efficient bilevel feasible solutions for the above problem are (0, 1, 0) and $\left(\frac{8}{5}, \frac{1}{5}, 0\right)$.

The efficient solutions for the problem (IQMBPP) are (0, 1, 0) and $\left(\frac{8}{5}, \frac{1}{5}, 0\right)$.

Indefinite Quadratic Bilevel Programming Problem with Multiple Objective Functions at Both Levels

In this section, we will be considering the indefinite quadratic bilevel program with multi-objective functions at both levels (IQBMPBL). It is defined as,

$$\begin{aligned} \text{(IQBMPBL):} \quad & \max_X (g_1(X,Y), g_2(X,Y), \cdots, g_s(X,Y)) \\ & \text{subject to } B_1^1 X + B_2^1 Y \le u^1 \\ & \text{where } Y \text{ solves} \\ & \max_Y (h_1(X,Y), h_2(X,Y), \cdots, h_t(X,Y)) \\ & \text{subject to } B_1^2 X + B_2^2 Y \le u^2 \\ & \qquad\qquad X, Y \ge 0, \end{aligned}$$

where $g_i(X,Y) = (p_{i1}X + q_{i1}Y + \gamma_i)\,(p_{i2}X + q_{i2}Y + \delta_i);\ i = 1, \cdots, s;$

$h_j(X,Y) = (a_{j1}X + b_{j1}Y + \eta_j)\,(a_{j2}X + b_{j2}Y + \varphi_j);\ j = 1, \cdots, t;$

Here, $X = \{x_1, , x_{n_3}\} \in \mathbb{R}^{n_3}$, $\quad Y = \{y_1, \cdots, y_{n_4}\} \in \mathbb{R}^{n_4}$

$p_{i1}, p_{i2} \in \mathbb{R}^{n_3};\ i = 1, \cdots, s$

$q_{i1}, q_{i2} \in \mathbb{R}^{n_4};\ i = 1, \cdots, s;\ \gamma_i,\ \delta_i \in \mathbb{R}\ ; i = 1, \cdots, s$

$a_{j1}, a_{j2} \in \mathbb{R}^{n_3};\ j = 1, \cdots, t;$

$b_{j1}, b_{j2} \in \mathbb{R}^{n_4};\ j = 1, ..., t;\ \eta_j,\ \varphi_j \in \mathbb{R};\ j = 1, \cdots, t$

$B_1^1 \in \mathbb{R}^{m_3 \times n_3}$; $B_2^1 \in \mathbb{R}^{m_3 \times n_4}$; $u^1 \in \mathbb{R}^{m_3}$

$B_1^2 \in \mathbb{R}^{m_4 \times n_3}$; $B_2^2 \in \mathbb{R}^{m_4 \times n_4}$; $u^2 \in \mathbb{R}^{m_4}$

Let $S_1 = \{(X, Y) : B_1^1 X + B_2^1 Y \le u^1;\ B_1^2 X + B_2^2 Y \le u^2;\ X,\ Y \ge 0\}$.

Here, the constraint region S_1 is a polyhedron for the problem (IQBMPBL). Also, S_1 is assumed to be non-empty and compact.

The lower level problem can be defined as,

(IQBMPBL (X_1)): $\underset{Y}{\text{Max}}(h_1(X_1, Y),\ h_2(X_1, Y), \cdots, h_t(X_1, Y))$

subject to $B_2^2 Y \le u^2 - B_1^2 X_1$

$Y \ge 0.$

The feasible region of the lower level problem is

$$T(X) = \{Y : B_2^2 Y \le u^2 - B_1^2 X;\ Y \ge 0)\}.$$

The inducible region of the upper level problem is

$\overline{T} = \{(X, Y) \in S_1 : Y \text{ solves } \underset{Y}{\text{Max}}(h_1(X, Y), \cdots, h_t(X, Y)), \text{ for a given } X\}.$

Again, $\overline{T}$ is assumed to be non-empty and the optimal solution of (IQBMPBL (X_1)) is singleton.

Algorithmic Development for (IQBMPBL) Problem

Here, S_1 is non-empty and compact set. The objective functions at both the levels, that is, g_i's $(i = 1, \cdots, s)$ at the upper level and h_j's $(j = 1, \cdots, t)$ at the lower level, are the product of two positive valued affine functions. Hence, the objective functions at both the levels are quasi-concave functions.

Consider the multi-objective problem at the upper level.

(IQBMPUL): $\underset{X}{\text{Max}}(g_1(X_1, Y),\ g_2(X_1, Y), \cdots, g_s(X, Y))$

subject to $(X, Y) \in S_1$.

Singe g_i's are quasi-concave functions, therefore, the optimal solution for each $g_i\ (i = 1, \cdots, s)$ exists at an extreme point. If this is a point of $\overline{T}$, then this is the extreme point of the inducible region, $\overline{T}$, [3].

Algorithm to Solve (IQBMPBL) Problem

Step 1 Consider the (IQBMPUL) problem, defined as

(IQBMPUL): $\underset{X}{\text{Max}}(g_1(X, Y),\ g_2(X, Y), \cdots, g_s(X, Y))$

subject to $(X, Y) \in S_1$.

Step 2 Set $i = 1$.

Let (X^i, Y^i) be an optimal solution of $g_i(X, Y)$, for each $i = 1, \cdots, s$.

Step 3 Put $X = X^i$ in the follower's problem, as

$$\underset{Y}{\text{Max}}\ (h_j(X^i, Y), j = 1, \cdots, t,$$
$$\text{subject to } B_2^2 Y \le u^2 - B_1^2 X^i$$
$$Y \ge 0.$$

Step 4 Set $j = 1$.
Let (X^i, Y_j) be an optimal solution of $h_j(X^i, Y)$.

Step 5.1 If $Y^i = Y_j$, then (X^i, Y^i) is a bilevel feasible solution of (IQBMPBL) problem.

Step 5.2 If $Y^i \neq Y_j$, set $j = j + 1$. Go to step 4 and repeat the process.

Step 5.3 If $Y^i \neq Y_j$, for any $j = 1, \cdots, t$; set $i = i + 1$. Go to step 2 and repeat the method.

Step 5.4 If no bilevel efficient feasible solution is obtained for any $i = 1, \cdots, s$ and for any $j = 1, \cdots, t$; find the next best solution of the first objective function, $g_1(X, Y)$. Go to Step 3 and repeat the procedure with $g_i(X, Y); i = 2, \cdots, s$.

Step 6 From the set of bilevel feasible solutions, find efficient solutions for the problem (IQBMPBL).

Example 2 : Consider the following indefinite quadratic bilevel programming problem with multi-objective functions at both levels,

(IQBMPBL):
$$\underset{x_1, x_2}{\text{Max}}\ g_1(x_1, x_2, x_3, x_4) = (2x_1 + x_2 + x_3 + 3)(x_2 + 2x_4 + 2)$$
$$\underset{x_1, x_2}{\text{Max}}\ g_2(x_1, x_2, x_3, x_4) = (x_1 + 2x_3 + x_4 + 4)(x_2 + x_3 + 2)$$
$$\text{subject to } 2x_1 + x_2 + x_3 \le 8$$
$$2x_2 + x_3 + x_4 \le 6$$
$$2x_1 + x_2 + x_4 \le 12$$

where (x_3, x_4) solves

$$\text{Max}\ h_1(x_1, x_2, x_3, x_4) = (2x_1 + x_3 + x_4 + 1)(x_2 + x_3 + x_4 + 3)$$
$$\text{Max}\ h_2(x_1, x_2, x_3, x_4) = (2x_1 + x_4 + x_3 + 2)(x_3 + x_2 + 4)$$
$$\text{subject to } -2x_1 + 4x_2 + x_3 + 2x_4 \le 16$$
$$x_2 + 2x_3 + x_4 \le 5$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

Solution: Using the algorithm, the bilevel feasible solutions of the above problem are,

(7/2, 1, 0, 4), (7/2, 0, 0, 5), (0, 1, 0, 4), (0, 0, 0, 5),
(11/4, 0, 5/2, 0), (13/6, 7/3, 4/3, 0), (0, 0, 5/2, 0), (25/6, 1, 0, 4)
(0, 7/3, 4/3, 0).

The set of efficient solutions for the problem (IQBMPBL) is

{(0, 0, 0, 5), (0, 0, 5/2, 0), (0, 7/3, 4/3, 0)}.

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ON

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OBJECTIVES

Optimization is a multi-disciplinary research field that deals with the characterization and computation of minima and / or maxima (local/global) of nonlinear, nonconvex, nonsmooth and discrete functions. optimization problems are frequently encountered in modeling of complex real world systems for a very broad range of applications including industrial and systems engineering, management science, operational research, mathematical economics, seismic optimization, production planning and scheduling, transportation and logistics and many other applied areas of science and engineering.

Due to rapid development and deployment of practical optimization methodologies in the last three decades, scientists around the globe in diverse disciplines have been using optimization techniques and algorithms to solve their respective problems. Optimization is playing a pivoting role in the development of modern science and engineering. The main aim of this conference is to provide impetus and motivation for further research work in all areas of optimization and its applications by providing a forum for the academic exchange of ideas and recent research works, both by senior and young active researchers from around the globe.

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All topics related to optimization theory, methods and applications, including, but not limited to:

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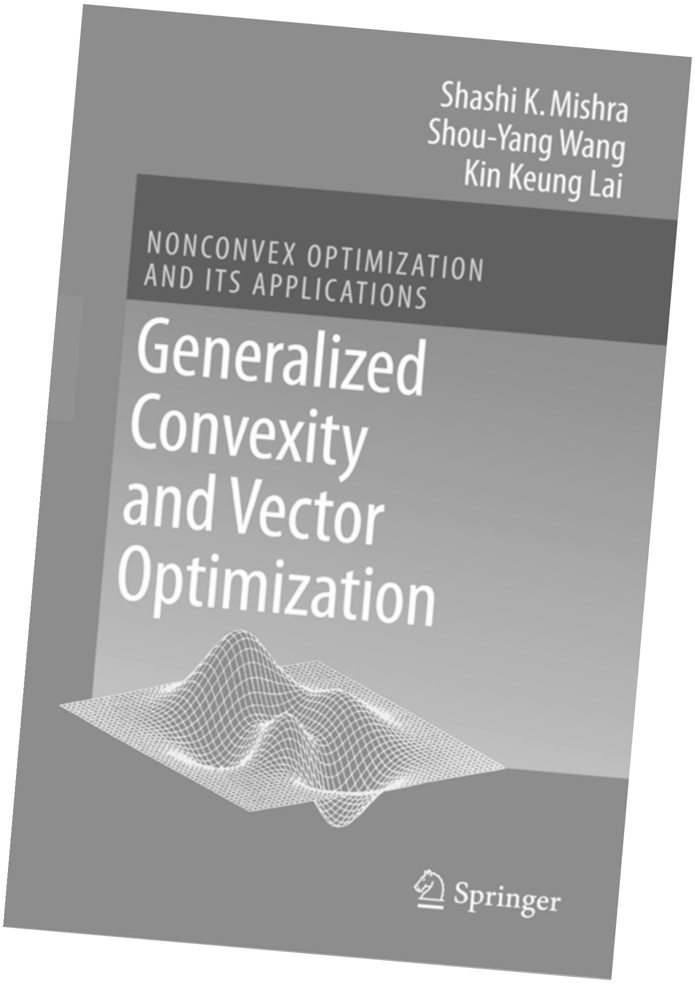

About this book

The present book discusses the Kuhn-Tucker Optimality, Karush-Kuhn-Tucker Necessary and Sufficient Optimality Conditions in presence of various types of generalized convexity assumptions. Wolfe-type Duality, Mond-Weir type Duality, Mixed type Duality for Multiobjective optimization problems such as Nonlinear programming problems, Fractional programming problems, Nonsmooth programming problems, Nondifferentiable programming problems, Variational and Control problems under various types of generalized convexity assumptions.

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V-INVEX FUNCTIONS AND VECTOR OPTIMIZATION summarizes and synthesizes an aspect of research work that has been done in the area of Generalized Convexity over the past several decades. Specifically, the book focuses on V-invex functions in vector optimization that have grown out of the work of Jeyakumar and Mond in the 1990's. V-invex functions are areas in which there has been much interest because it allows researchers and practitioners to address and provide better solutions to problems that are nonlinear, multi-objective, fractional, and continuous in nature. Hence, V-invex functions have permitted work on a whole new class of vector optimization applications.

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