Unifying Efficiency and Weak Efficiency in Generalized Quasiconvex Vector Minimization on the Real-line

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Abstract. Usually the concepts of efficient and weakly efficient solution are associated to a multiobjective optimization problem. Both notions may be described in terms of a preference relation determined by a closed convex cone with nonempty interior - typically the nonnegative orthant of a finite dimensional space. We present a unified approach for dealing with both notions at the same time under generalized quasiconvexity assumptions on the objectives defined on the real-line. Since most algorithms in scalar minimization involve the solvability of a one-dimensional problem to find the next iterate, it is expected that our results be applied in the vector case.

We established several characterizations of the nonemptiness of the solution set, and also various characterizations when besides boundedness is required. To that end we used a notion of relaxed convexity for vector functions introduced earlier by one of the authors.

Keywords. nonconvex vector optimization, quasiconvex vector functions, efficiency, weak efficiency, generalized convexity.

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1 Introduction and Formulation of the Problem

In multiobjective optimization several criterion functions must be minimized simultaneously. Very often, no single point minimizing all criteria at once may be found, and so the concept of optimality has to be modified. Usually, the notion of efficient or weakly efficient solution is considered. A point is called efficient or Pareto-optimal, if there does not exist a different point with smaller or equal objective function values, such that there is a decrease in at least one objective function value; a point is called weakly efficient or weakly Pareto-optimal, if there exists no other point with strictly smaller objective function value. Both notions may be described in terms of a preference relation determined by a closed convex cone with nonempty interior - typically the nonnegative orthant of some $\mathbb{R}^m$.

More precisely, given a nonempty set $K \subseteq \mathbb{R}^n$, a closed convex cone $P \subseteq \mathbb{R}^m$, and a vector-valued function, $F : K \rightarrow \mathbb{R}^n$, a point $\bar{x} \in K$ is an efficient solution to the problem

$$
\min \{F(x) : x \in K\}
$$

if

$$
F(x) - F(\bar{x}) \notin -P \setminus l(P) \quad \forall x \in K,
$$

where $l(P) = P \cap (-P)$; whereas when $\text{int } P \neq \emptyset$, $\bar{x}$ is a weakly efficient solution to (1) if

$$
F(x) - F(\bar{x}) \notin -\text{int } P \quad \forall x \in K.
$$

We denote by $E = E(K)$ the set of efficient solutions to (1), and by $E_w = E_w(K)$ the set of weakly efficient solutions. Since $P \subseteq \mathbb{R}^m \setminus (-P \setminus l(P)) \subseteq \mathbb{R}^m \setminus (-\text{int } P)$, every efficient solution is also weakly efficient. We refer to [24, 25, 19] for a theoretical treatment of even more general vector optimization problems.

Problem (3) has been extensively studied mainly when $P = \mathbb{R}^m_+$; among the recent publications for the existence of weakly efficient solutions, we quote [7, 13, 15, 16, 17, 24, 27, 9] and references therein; whereas results about existence of efficient solutions may be found in [10, 12, 19, 25]. The case $K \subseteq \mathbb{R}$ is of particular interest since it is known that most algorithms in scalar minimization (e.g. Newton, gradient, projection methods) require to solve a one-dimensional minimization problem. It is expected that similar devices be applied in multiobjective optimization. Thus, purpose of this work is to give a unified approach for both problems in the particular case when $K \subseteq \mathbb{R}$ under generalized quasiconvexity assumptions, so we will be interested in the problem:

$$
\text{find } \bar{x} \in K : F(x) - F(\bar{x}) \in S \quad \forall x \in K,
$$

where $S$ is a cone satisfying $S + P \subseteq S$. This inclusion is satisfied if either $S = \mathbb{R}^m \setminus (-P \setminus l(P))$ or $S = \mathbb{R}^m \setminus (-\text{int } P)$.

Although in recent years nonlinear-scalarizations schemes are being employed in multiobjective optimization problems-specially in the absence of the standard convexity assumptions ([11])-the linear-scalarization tool, or weighting method, is among the main procedures, even in the quasiconvexity
However, the main drawback lies on the choice of the parameters since it is not known in advance which ones give rise to solutions. In fact, bad choices of these parameters can lead to unbounded scalar optimization problems, or under convexity, this method computes only *proper* efficient solutions [21].

The notion of semistrict \((S)\)-quasiconvexity for vector functions, suitable for studying Problem (4), is recalled in Section 2, where it is also established some relationship with other classes of functions. Section 3 provides several characterizations of the nonemptiness of the solution set to (4), and also various necessary and sufficient conditions when in addition boundedness is required. Several examples are presented in each section showing our results are in some sense optimal.

We end this section by recalling the definition of asymptotic cone. Given a set \(C\), the asymptotic cone of \(C\), denoted by \(C^\infty\), is defined by

\[
C^\infty = \left\{ v \in \mathbb{R}^n : \exists t \downarrow 0, \exists \lambda \in C, t \lambda \to v \right\},
\]

which is a closed cone. The “recession” term instead of asymptotic is employed when convex sets are considered. Some of the main properties of asymptotic cones may be found in [26, Chapter 3].

**Proposition 1.1** The following assertions hold.

(a) \(K_1 \subseteq K_2\) implies \(K_1^\infty \subseteq K_2^\infty\);

(b) let \(K \subseteq \mathbb{R}^n\), then \(K\) is bounded if and only if \(K^\infty = \{0\}\);

(c) Let \(\{K_i\}_{i \in I}\) be any family of nonempty subsets of \(X\), then

\[
\left( \bigcap_{i \in I} K_i \right)^\infty \subseteq \bigcap_{i \in I} (K_i)^\infty.
\]

If, in addition, \(\bigcap_{i \in I} K_i \neq \emptyset\) and each set \(K_i, i \in I\), is closed and convex, then we obtain an equality in the previous inclusion.

### 2 Semistrict \((S)\)-quasiconvexity and Related Properties

Let \(X, Y\) be real normed vector spaces. We are also given a nonempty set \(S \subseteq Y\), a nonempty convex set \(K \subseteq X\), and a mapping \(F : K \to Y\). It is requested to find

\[
\overline{x} \in K : F(y) - F(\overline{x}) \in S, \forall y \in K.
\]  (5)

A point \(\overline{x} \in K\) satisfying (5) is called a (global) \(S\)-minimal of \(F\) (on \(K\)) and the set of such points is denoted by \(E_S\). Obviously \(E_S \neq \emptyset\) implies that \(0 \in S\).

In connection to problem (5) the following definition introduced in [15], and further developed in [16, 17], will play an important role. In the following: “\(\text{co}(A)\)” stands for the convex hull of the set \(A\), which is the smallest convex set containing \(A\); given \(x \neq y\), we set

\[
[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}.
\]
**Definition 2.1.** The vector function $F : K \rightarrow Y$ is said to be:

(i) semistrictly $(S)$-quasiconvex at $y \in K$, if for every $x \in K, x \neq y$,
\[
F(x) - F(y) \in -S \Rightarrow F(\xi) - F(y) \in -S \quad \forall \xi \in ]x, y[.
\]

We say that $F$ is semistrictly $(S)$-quasiconvex (on $K$) if it is at every $y \in K$.

(ii) explicitly $(S)$-quasiconvex at $y \in K$, if it is semistrictly $(S)$-quasiconvex and semistrictly $(Y \setminus S)$-quasiconvex at $y$.

We say that $F$ is explicitly $(S)$-quasiconvex (on $K$) if it is at every $y \in K$.

When $Y = \mathbb{R}$, the previous definition reduces to quasiconvexity of real-valued functions in case $S = \mathbb{R}_+ = [0, \infty)$, and to semistrict quasiconvexity if $S = \mathbb{R}_+ = (0, \infty)$.

In what follows, $P \subseteq Y$ is a (not necessarily pointed) convex cone, $P \neq Y$. The set $P' \subseteq Y'$, with $Y'$ being the topological dual of $Y$, is the (nonnegative) polar cone defined by
\[
P' = \{ p' \in Y' : \langle p', p \rangle \geq 0 \quad \forall \ p \in P \},
\]
where $\langle \cdot,\cdot \rangle$ stands for the duality pairing between $Y$ and $Y'$.

We recall two important facts if additionally $P$ is also closed: the first one is the bipolar theorem ($P'' = P$),
\[
p \in P \iff \langle p', p \rangle \geq 0 \quad \forall \ p' \in P',
\]
and in case $\text{int} \ P \neq \emptyset$, we have
\[
p \in \text{int} \ P \iff \langle p', p \rangle > 0 \quad \forall \ p' \in P' \setminus \{0\}.
\]

Some generalizations of convexity in connection with optimality conditions are provided in [5, 6, 22].

The next definition collects some of the main notions of generalized convexity employed in the existence theory of vector optimization problems appearing in the literature.

**Definition 2.2** Let $K \subseteq X$ be a convex set and $P$ be a convex cone. The function $F : K \rightarrow Y$, is said to be

(i) $P$-convex if for all $x, y \in K$,
\[
\alpha F(x) + (1 - \alpha)F(y) \in F(\alpha x + (1 - \alpha)y) + P \quad \text{for all } \alpha \in (0,1).
\]

In particular, $F$ is $\mathbb{R}_+^n$-convex if and only if each component of $F$ is convex;

(ii) properly $P$-quasiconvex ([12]) if for each $x, y \in K$,
\[
F([x, y]) \subseteq \{ F(x), F(y) \} - P,
\]
or equivalently, the set
\[
\{ \xi \in K : F(\xi) \not\in \lambda + P \} \text{ is convex } \text{ for all } \lambda \in Y.
\]
(iii) naturally \( P \)-quasiconvex ([28]), if for each \( x, y \in K \),
\[
F([x, y]) \subseteq [F(x), F(y)] - P.
\]

(iv) scalarly \( P \)-quasiconvex ([23]) if
\[
x \in K \mapsto \{p^*, F(x)\} \text{ is quasiconvex for every } p^* \in P^*.
\]

(v) \( P \)-quasiconvex ([12, 24]) if the set
\[
\{\xi \in K : F(\xi) \in \lambda - P\} \text{ is convex for all } \lambda \in Y.
\]

In particular, \( F \) is \( \mathbb{R}^n \)-quasiconvex if and only if each component of \( F \) is quasiconvex;

(vi) \( (\text{int } P \neq \emptyset) \) [15, 16, 17] semistrictly \((Y - \text{int } P)\)-quasiconvex (see Definition 2.1, [15]), if for every \( x, y \in K \),
\[
F(x) - F(y) \notin \text{ int } P \Rightarrow F(\alpha x + (1-\alpha)y) - F(y) \notin \text{ int } P \forall \alpha \in (0,1).
\]

(vii) semistrictly \((Y \setminus (P \setminus \text{int } P))\)-quasiconvex (see Definition 2.1) if for every \( x, y \in K \),
\[
F(x) - F(y) \notin \text{ P \setminus \text{int } P} \Rightarrow F(\alpha x + (1-\alpha)y) - F(y) \notin \text{ P \setminus \text{int } P} \forall \alpha \in (0,1).
\]

In general, as noted in [12, 15], there is no relationship between the notions of \( P \)-convexity and proper \( P \)-quasiconvexity.

Also with no further assumption on \( P \), the class of naturally \( P \)-quasiconvex functions is strictly larger than the preceding two classes of functions as shown in [28]. On the other hand, the class of \( P \)-quasiconvex functions is strictly larger than the scalarly \( P \)-quasiconvex functions as shown in [23] (where the terminology *-quasiconvexity is applied), and this latter class of functions is equivalent to the class of naturally \( P \)-quasiconvex functions as established in Theorem 2.3 below.

Furthermore, the function \( F(x) = (\frac{1}{|x||}, |x|), x \in \mathbb{R}, \) is semistrictly \((\mathbb{R}^2 - \text{int } \mathbb{R}^2_+)\)-quasiconvex but not \( \mathbb{R}^2_+ \)-quasiconvex.

We also notice that there is no relationship between (vi) and (vii), and also between (v) and (vii).

Indeed, by taking \( K = [0, 1] \), the function \( F = (f_1, f_2) \) given by
\[
f_1(x) = \begin{cases} -x + 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}, \quad f_2(x) = x,
\]
is semistrictly \((\mathbb{R}^2 \setminus (\mathbb{R}^2_+ \setminus \{0\}))\)-quasiconvex without being semistrictly \((\mathbb{R}^2 - \text{int } \mathbb{R}^2_+)\)-quasiconvex
(and therefore neither \( \mathbb{R}^2_+ \)-quasiconvex, because of Theorem 2.3 below) since \( F(1) - F(0) = (0,1) \notin \text{ int } \mathbb{R}^2_+ \) and \( F(1/2) - F(0) = (1/2, 1/2) \notin \text{ int } \mathbb{R}^2_+ \); whereas the function
\[
f_1(x) = x, \quad f_2(x) = \begin{cases} 0 & \text{if } 0 \geq 1 \\ 1 & \text{if } 0 \leq x < 1 \end{cases}, \quad K = [0, +\infty),
\]
is \( \mathbb{R}^2_+ \)-quasiconvex (thus semistrictly \((\mathbb{R}^2 - \text{int } \mathbb{R}^2_+)\)-quasiconvex by the next theorem), but not semistrictly \((\mathbb{R}^2 \setminus (\mathbb{R}^2_+ \setminus \{0\}))\)-quasiconvex at 0 since \( F(1) - F(0) = (1, -1) \notin \mathbb{R}^2_+ \setminus \{0\} \) and \( F(1/2) - F(0) = (1/2, 0) \notin \mathbb{R}^2_+ \setminus \{0\} \).
Certainly, proper $P$-quasiconvexity and $P$-quasiconvexity are the more common generalizations of the notion of quasiconvexity for real-valued functions. The semistrict $(Y \setminus \text{int } P)$-quasiconvexity and the semistrict $(Y \setminus (P \setminus l(P)))$-quasiconvexity provide further generalizations as well: the first being suitable when dealing with weak efficiency ([15, 16, 17]) and the second for efficiency.

The next theorem establishes the relationship between the notions introduced in Definition 2.2.

**Theorem 2.3** Assume $K \subseteq X$ is a convex set and $P \subseteq Y$ a convex cone such that $P \neq Y$. Concerning Definition def-conv-gen, we have the following assertions: itemize

(a) $(i) \Rightarrow (iii) \Rightarrow (v)$, $(ii) \Rightarrow (iii) \Rightarrow (iv)$, $(iii) \Rightarrow (vi)$, $(iii) \Rightarrow (vii)$;

(b) if in addition $P$ is closed then

$(iii) \iff (iv) \Rightarrow (v) \Rightarrow (vi)$.

(c) if additionally $P \cup (-P) = Y$ then,

$(v) \iff (ii) \iff (iii)$ and $(v) \Rightarrow (iv)$;

(d) if additionally $P \cup (-P) = Y$ and $P$ is closed (and so $\text{int } P \neq \emptyset$), then

$(ii) \iff (iii) \iff (iv) \iff (v) \iff (vi) \iff (vii)$.

**Proof:** (a): The implication $(iii) \Rightarrow (vii)$ follows from Proposition 2.6 below with $S = Y \setminus (P \setminus l(P))$; the other implications appear in Theorem 2.6 of [15] since they are still valid in general normed vector spaces.

(b): When $\text{int } P \neq \emptyset$, the equivalence is proved in Proposition 3.9 of [18]. However, the proof still remains valid in the general case with obvious changes. We reproduce the proof for reader's convenience.

Assume that $F$ satisfies $(iv)$. We need to check that given $t \in R$ and $x^* \in P^*$, the set $K_t = \{ z \in K : \langle x^*, F(z) \rangle \leq t \}$ is convex. Indeed, if $x, y \in K$, then by natural-$P$-quasiconvexity of $F$, for all $z \in [x, y]$ there exists $\lambda \in [0, 1]$ and $u \in P$ such that $F(z) = \lambda F(x) + (1 - \lambda) F(y) - u$. Hence,

$$\langle x^*, F(z) \rangle = \lambda \langle x^*, F(x) \rangle + (1 - \lambda) \langle x^*, F(y) \rangle - \langle x^*, u \rangle \leq t$$

thus $z \in K_t$, so $K_t$ is convex.

Conversely, assume that $F$ is not naturally-$P$-quasiconvex. Then, there exist $x, y \in K$ and $z \in [x, y]$ such that for all $\mu \in [0, 1]$, $F(z) \notin \mu F(x) + (1 - \mu) F(y) - P$. Thus, by (6), for every $\mu \in [0, 1]$ there exists $x^* \in P^* \setminus \{0\}$ such that

$$\langle x^*, F(z) \rangle \geq \langle x^*, \mu F(x) + (1 - \mu) F(y) \rangle.$$ 

We can suppose that $\| x^* \| = 1$. By Alaoglu’s theorem (see [1] or [Theorem 5.25][2]), the set $B = \{ y^* \in Y^* : \| y^* \| \leq 1 \}$ is weak-star compact. Thus,

$$x^* \in B \cap P^* = \{ y^* \in P^* : \| y^* \| \leq 1 \}$$
is also weak-star compact. Setting $f(y', \mu) = \{ y', F(z) - \mu F(x) - (1 - \mu) F(y) \}$, we can apply the Sion minimax theorem (see for instance [23]) to get

$$\max_{y \in K_x} \min_{\mu \in [0, 1]} f(y', \mu) = \min_{\mu \in [0, 1]} \max_{y \in K_x} f(y', \mu) > 0.$$ 

Hence there exists $x^* \in B_y$ such that

$$\left\langle x^*, F(z) \right\rangle > \mu \left\langle x^*, F(x) \right\rangle + (1 - \mu) \left\langle x^*, F(y) \right\rangle \quad \forall \mu \in [0, 1].$$

In particular, we get $x^* \in P'$ and $\left\langle x^*, F(z) \right\rangle > \left\langle x^*, F(x) \right\rangle$ and $\left\langle x^*, F(z) \right\rangle > \left\langle x^*, F(y) \right\rangle$.

Thus $F$ is not scalarly $P$-quasiconvex.

For the other implications we refer Theorem 2.6 of [15].

(c): It may be found in Theorem 2.6 of [15].

(d): This is a consequence of (a), (b) and (c) and the fact that $P = Y \setminus \text{int} P$ and $P \setminus \text{int} P = \text{int} P$.

Notice that in this case there exists $y^* \in P' \setminus \{0\}$ such that $P = \{ p \in Y : \left\langle p^*, p \right\rangle \geq 0 \}$ (see, for instance, [15]).

For a given $y \in K$, we set

$$S_y = \{ x \in K : F(x) - F(y) \in -S \}.$$ 

The proof of the next lemma follows immediately from Definition 2.1.

**Lemma 2.4** [16] Given $S, K$ as above and $F : K \to Y$. For fixed $y \in K$, the following two assertions are equivalent: itemize

(a) $F$ is semistrictly $(S)$-quasiconvex at $y$;

(b) $[x, y] \subseteq S_y$ for all $x \in S_y$.

We recall that $C$ is starshaped at $y \in C$, if $[x, y] \subseteq C$, for all $x \in C$.

Thus, if $C$ is starshaped at $y \in C$, then

$$C^- = \bigcap_{t > 0} t(C - y). \quad \text{(8)}$$

In case $0 \in S$, we have $y \in S_y$ for all $y \in K$, and therefore, we obtain the following theorem without any continuity assumption.

**Theorem 2.5** Given $S, K$ as above with $0 \in S$, and $F : K \to Y$. The following assertions hold.

(a) $F$ is semistrictly $(S)$-quasiconvex at $y \in K$ if, and only if, $S_y$ is starshaped at $y$;

(b) Assume, in addition that $K$ is closed. If $F$ is semistrictly $(S)$-quasiconvex at $y$ then

$$(S_y)^- = \{ v \in K^- : F(y + \lambda v) - F(y) \in -S \quad \forall \lambda > 0 \}.$$ 

**Proof:** (a) follows from the previous lemma. Part (b) is a consequence of (8), since

$$v \in (S_y)^- \iff y + tv \in S_y, \quad \forall t > 0 \iff v \in K^-, F(y + tv) - F(y) \in -S \quad \forall t > 0.$$ 

**Proposition 2.6** Let $0 \neq S \subseteq Y$ such that $tS \subseteq S$ for all $t > 0; K \subseteq X$ be convex; $P \subseteq Y$ be a convex cone satisfying $S + P \subseteq S$. Let $F : K \to Y$ be given.
(a) If $F$ is $P$-convex, then $F$ is explicitly $(S)$-quasiconvex.
(b) Assume that $0 \in S$ and that $F$ is naturally $P$-quasiconvex, then $F$ is semistrictly $(S)$-quasiconvex.

**Proof:** (a): Let $x, y \in K$ and $x \neq y$. By $P$-convexity we have $F(tx+(1-t)y) \in tf(x)+(1-t)f(y)$ for all $t \in (0,1)$. Thus, by setting $\xi_t = tx+(1-t)y$, we have $F(\xi_t) - F(y) \in t(F(x) - F(y)) - P$ for all $t \in (0,1)$. If $F(x) - F(y) \in -S$, then $F(\xi_t) - F(y) \in -tS - P \subseteq -S$, proving the semistrict $(S)$-quasiconvexity. On the other hand, if $F(x) - F(y) \in Y \setminus S$ then $F(\xi_t) - F(y) \in t(Y \setminus S) - P \subseteq Y \setminus S$. It proves the semistrict $(Y \setminus S)$-quasiconvexity, which completes the proof.

(b): Given $x, y \in K$, we have that for all $\xi \in [x,y]$,

$$F(\xi) \in [F(x), F(y)] - P.$$ 

Thus, for every $\xi \in [x,y]$ there exists $\mu \in [0,1]$ such that $F(\xi) - F(y) \in \mu(F(x) - F(y)) - P$. If $F(x) - F(y) \in -S$, then $F(\xi) - F(y) \in -\mu S - P \subseteq -S - P \subseteq -S$, proving the desired result.

The class of explicit $(S)$-quasiconvexity also includes that of explicit quasiconvexity componentwise when $P = R^m_+$ and $S = Y \setminus \text{int} R^m_+).$ More generally, we say that $F: K \rightarrow Y$ is **explicitly scalarly $P$-quasiconvex** if it is scalarly $P$-quasiconvex (see (iv) in Definition 2.2) and semistrictly scalarly $P$-quasiconvex: the latter means, that

$$x \in K \mapsto \{ p^*, F(x) \}$$

is semistrictly quasiconvex for every $p^* \in P^*$.

**Proposition 2.7**: Let $\emptyset \neq K \subseteq X$ be convex and closed. Assume that $P \subseteq Y$ is a closed, convex cone. If $F: K \rightarrow Y$ is explicitly scalarly $P$-quasiconvex then it is explicitly $(P \setminus \text{int}(P))$-quasiconvex, and explicitly $(\text{int}(P))$-quasiconvex (here $\text{int} P \neq \emptyset$).

**Proof:** Let $x, y \in K$. If $F(x) - F(y) \in -(P \setminus \text{int}(P))$, then $\langle p^*, F(x) - F(y) \rangle \leq 0$ for all $p^* \in P^*$, and $\langle q^*, F(x) - F(y) \rangle < 0$ for some $q^* \in P^*$. By hypothesis, $\{ p^*, F(\xi) - F(y) \} \leq 0$ and $\langle q^*, F(\xi) - F(y) \rangle < 0$ for all $\xi \in [x,y]$. Hence, $F(\xi) - F(y) \in P \setminus \text{int}(P)$ for all $\xi \in [x,y].$

Assume that $F(x) - F(y) \in P \setminus \text{int}(P)$. If on the contrary, $F(\xi) - F(y) \in P \setminus \text{int}(P)$ for some $\xi \in [x,y]$, then $\langle p^*, F(\xi) - F(y) \rangle \geq 0$ for all $p^* \in P^*$ and $\langle q^*, F(\xi) - F(y) \rangle > 0$ for some $q^* \in P^*$. Thus, $\{ p^*, F(x) - F(y) \} \geq 0$ for all $p^* \in P^*$ and $\langle q^*, F(x) - F(y) \rangle > 0$. Hence, $F(x) - F(y) \in P \setminus \text{int}(P)$, a contradiction. This completes the proof that $F$ is explicitly $(P \setminus \text{int}(P))$-quasiconvex.

The explicit $(\text{int}(P))$-quasiconvexity of $F$ follows from Theorem 5.3 of [15].

When $P^*$ is the weak-star closed convex hull of its extreme directions, one obtains part of the previous proposition under weaker assumptions. In what follows, extrd $P^*$ stands for the set of extreme directions of $P^*$: here $q^* \in \text{extrd} P^*$ if and only if $q^* \in P^* \setminus \{0\}$ and for all $q^*_1, q^*_2 \in P^*$ such that $q^* = q^*_1 + q^*_2$ we actually have $q^*_1, q^*_2 \in R_+q^*$.

**Proposition 2.8**: Let $\emptyset \neq K \subseteq X$ be convex and closed, $P \neq Y$ be a closed convex cone in $Y$ and $F: K \rightarrow Y$ be given.

(a) Assume that $P^*$ is the weak-star closed convex hull of extrd $P^*$. If for all $p^* \in \text{extrd} P^*$,

$$x \in K \mapsto \{ p^*, F(x) \}$$

is quasiconvex and semistrictly quasiconvex,
then it is explicitly \((P \setminus l(P))\)-quasiconvex. If, in addition \(P^\star\) is polyhedral with \(\text{int } P \neq 0\), then \(F\) is also explicitly \((\text{int } P)\)-quasiconvex.

(b) Let \(D = \{p^\star \in P : \|p^\star\| = 1\}\).

(b1) If for all \(p^\star \in D,\)

\[
x \in K \mapsto \langle p^\star, F(x) \rangle
\]

is quasiconvex and semistrictly quasiconvex,

then \(F\) is explicitly \((P \setminus l(P))\)-quasiconvex.

(b2) \((\text{int } P = 0)\) If for all \(p^\star \in D,\)

\[
x \in K \mapsto h_{p^\star}(x) = \langle p^\star, F(x) \rangle
\]

is quasiconvex,

then \(F\) is semistrictly \((Y \setminus \text{int } P)\)-quasiconvex.

(b3) If for all \(p^\star \in D, h_{p^\star}\) is semistrictly quasiconvex, then \(F\) is semistrictly \((\text{int } P)\)-quasiconvex.

**Proof:** (a): It follows the same argument of the previous proposition and using the following two facts (see (6)):

\[
p \in P \iff \langle p^\star, p \rangle \geq 0 \quad \forall p^\star \in \text{extr } P^\star, \quad (9)
\]

and if for some \(z \in Y\) there is \(p^\star \in P^\star\) such that \(\langle p^\star, z \rangle < 0\), then there is \(q^\star \in \text{extr } P^\star\) satisfying \(\langle q^\star, z \rangle < 0\). The last assertion follows easily.

We apply a similar reasoning as that in the preceding proposition to prove Parts (b1), (b2) and (b3).

Other sufficient conditions ensuring explicit \((\text{int } P)\)-quasiconvexity may be found in [15].

**Remark 2.9** Conditions ensuring that \(P^\star\) is the weak-star closed convex hull of \(\text{extr } P^\star\), may be found in Remark 2.2 of [3]. It is true, in particular, when \(\text{int } P \neq 0\). In that paper was also proved that \(P\)-quasiconvexity is equivalent to the quasiconvexity of

\[
x \in K \mapsto \langle p^\star, F(x) \rangle \quad \text{for every } p^\star \in \text{extr } P^\star.
\]

When \(P = \mathbb{R}^n_+\), \(\text{extr } \mathbb{R}^n_+\) reduces to the canonical basis of \(\mathbb{R}^n\). Thus, for any vector function the assumptions in the previous theorem read by saying that each of its components is quasiconvex and semistrictly quasiconvex.

**Remark 2.10** By virtue of Remark 2.9, (b2) of Proposition 2.8 remains true if \(D\) is replaced by \(\text{extr } P^\star\); whereas (b3) continues to be valid as above, under the stronger assumption \(P^\star = \text{co}(\text{extr } P^\star)\), which is satisfied if \(P\) is polyhedra or if \(P^\star\) is a kind of ice-cream cone.

**Example 2.11** This example shows that explicit \((\mathbb{R}^n_+ \setminus \{0\})\)-semistrict quasiconvexity does not imply the semistrict quasiconvexity of each component. Take \(K = [0, +\infty), P = \mathbb{R}^2_+\) and \(F = (f_1, f_2)\) with

\[
f_1(x) = x, \quad f_2(x) = \begin{cases} 
0 & \text{if } x = 0, \\
1 & \text{if } x \neq 0.
\end{cases}
\]

Here, \(E = \{0\}\) and \(E_w = \{0\}\).
3 Unifying Efficiency and Weak Efficiency on the Real-line: the Main Results

We are concerned in the problem

\[
\text{find } \bar{x} \in K : F(y) - F(\bar{x}) \in S \quad \forall y \in K.
\] (10)

The following hypothesis will be needed in the sequel.

**Hypothesis (A)**: Problem (10) admits a solution for all compact convex set \( K \subseteq X \).

Concerning the validity of Hypothesis (A), we have the next lemma.

**Lemma 3.1** Let \( X, Y \) be normed vector spaces, \( K \subseteq X \) compact, \( P \subseteq Y \) be convex cone such that \( P \neq Y \), and \( F : K \to Y \) be given. Then, Hypothesis (A) holds under any of the following circumstances:

(a) (\cite[Theorem 5.1]{Theorem}) \( S = Y \setminus (-P \setminus l(P)) \) and the set \( \{x \in K : F(x) - F(y) \in -P \} \) is closed for all \( y \in K \);

(b) (\cite[Theorem 3.2]{Theorem}) \( S = Y \setminus \text{int } P, \text{ int } P \neq \emptyset \) and the set \( \{x \in K : F(x) - F(y) \notin \text{ int } P \} \) is closed for all \( y \in K \).

It is known from \cite{Reference} (see also \cite{Reference2,Reference3}) that if \( F \) is \( P \)-lower semicontinuous then (a) and (b) hold.

We recall that (\cite{Reference4}) \( F : K \to Y \) is \( P \)-lower semicontinuous (\( P \)-lsc) at \( x_0 \in K \) if for any open set \( V \subseteq Y \) such that \( F(x_0) \in V \) there exists an open neighborhood \( U \subseteq X \) of \( x_0 \) such that \( F(U \cap K) \subseteq V + P \). We shall say that \( F \) is \( P \)-lsc (on \( K \)) if it is at every \( x_0 \in K \).

We point out that \( F = (f_1, \ldots, f_m) \) is \( \mathbb{R}^n_+ \)-lsc if and only if each \( f_i \) is lsc.

In connection to Problem (4) and following the same line of reasoning as in \cite{Reference5,Reference6,Reference7}, we set

\[
R_s = \bigcap_{y \in K} \left\{ v \in K^\circ : F(y + \lambda v) - F(y) \in -S \quad \forall \lambda > 0 \right\}.
\] (11)

From now on, we restrict ourselves to the case \( X = \mathbb{R} \). Notice that \( P \)-lower semicontinuity of \( F \) is not assumed in the next two theorems.

**Theorem 3.2** Let \( \emptyset \neq K \subseteq X \) be closed and convex, let \( \emptyset \neq S \subseteq Y \) such that \( 0 \in S \), and \( F : K \to Y \) be semistrictly (\( S \))-quasiconvex. Assume that \( E_s \neq \emptyset \), then

\[
(E_s)^\circ \subseteq R_s.
\] (12)

If, in addition, every \( S_y \) is closed and convex for all \( y \in K \), then \( (E_s)^\circ = R_s \).

**Proof:** If \( E_s \neq \emptyset \) then

\[
\emptyset \neq (E_s)^\circ = \bigcap_{y \in K} (S_y)^\circ \subseteq \bigcap_{y \in K} (S_y)^\circ = R_s,
\]

by Theorem 2.5. The last part is a consequence of the previous inclusion by taking into account Proposition 1.1.

The closedness of \( S_y \) is satisfied if \( F \) is \( P \)-lsc; and the convexity of each \( S_y \) holds if either \( X = \mathbb{R} \) or \( Y = \mathbb{R} \) (in such a case \( P = \mathbb{R}_+ \)) provided \( F \) is semistrictly (\( S \))-quasiconvex. However, when \( X = \mathbb{R} \), the same result is obtained without the closedness of \( S_y \), as Theorem 3.4 below shows.
We now exhibit some instances where the inclusion in (12) does not hold without the semistrict \((S)\)-quasiconvexity assumption. The first example concerns efficiency and the second weak efficiency.

**Example 3.3** We really need the semistrict \((S)\)-quasiconvexity in Theorem 3.2.

(i) Take \(K = [0, +\infty), S = \mathbb{R}^2 \setminus \left(\mathbb{R}_+^2 \setminus \{0\}\right)\) and

\[
f_1(x) = \begin{cases} x-1, & \text{if } x \in [0, 2), \\ x, & \text{if } x \in [2, +\infty).
\end{cases}
\]

\[
f_2(x) = \begin{cases} 4, & \text{if } x \in (-\infty, 0], \\ 6-x, & \text{if } x \in (2, +\infty).
\end{cases}
\]

An easy computation shows that \(\{0\}\) and \(\{0\}\) whereas \(\{0\}\).

(ii) Take \(K = \mathbb{R}, S = \mathbb{R}^2 \setminus \text{int } \mathbb{R}^2\) and

\[
f_1(x) = \begin{cases} x, & \text{if } x \in (-\infty, 2], \\ x-4, & \text{if } x \in (2, +\infty).
\end{cases}
\]

\[
f_2(x) = \begin{cases} 4, & \text{if } x \in (-\infty, 0], \\ 4+x, & \text{if } x \in [0, 2), \\ x-5, & \text{if } x \in (2, +\infty).
\end{cases}
\]

Here \(E_w = E_s = (-\infty,0] \cup \{4\}, (E_s)^c = (-\infty,0] \) and \(R_s = \{0\}\). Setting \(F = (f_1, f_2)\), we observe that \(F(0) - F(3) = (-1,-1) \) and \(F(2) - F(3) = (1,1) \) \(\in \mathbb{R}^2\). Theorem 3.4

Let \(0 \neq K \subseteq \mathbb{R}\) be closed and convex, \(0 \neq S \subseteq Y\) such that \(0 \in S\), and \(F : K \to Y\) be semistrictly \((S)\)-quasiconvex. Assume that \(E_s \neq 0\), then \(F\) is convex and \((E_s)^c = R_s\).

**Proof:** By virtue of the first part of the previous theorem, we need only to prove that \(R_s \subseteq (E_s)^c\). Take \(0 \neq v \in R_s\). We consider \(v > 0\) (the case \(v < 0\) is analyzed in a similar way). Take any \(x \in E_s\), we will check that \(x + \lambda v \in E_s\) for all \(\lambda > 0\). Let \(y \in K\) and \(\lambda > 0\). If \(x + \lambda v \in \text{co}(\{x, y\})\), then \(F(x + \lambda v) - F(y) \in -S\) by assumption on \(F\). If \(x + \lambda v \in \text{co}(\{x, y\})\), we distinguish the cases \(x < y\) and \(y < x\). If \(x < y\), then \(x + \lambda v \in (y, y + \lambda v)\). Therefore \(F(y) - F(x + \lambda v) \in -S\). If \(y < x < y + \lambda v\), then by writing \(x = y + \rho v\) for some \(\rho > 0\), we obtain \(F(x + \lambda v) - F(y) = F(y + \rho v) - F(y) \in -S\). Hence \(x + \lambda v \in E_s\) for all \(\lambda > 0\), which implies that \(v \in (E_s)^c\), and thus the proof is completed.

One can find examples in higher dimension where the convexity of \(E_w\) fails under the standard convexity assumption on \(F\), see for instance [13].

**Theorem 3.5** Let \(0 \neq K \subseteq \mathbb{R}\) be a closed convex set, let \(S \subseteq Y\) be given. Assume that \(F : K \to Y\) is semistrictly \((S)\)-quasiconvex. The following assertions are equivalent:

(a) \(R_s = \{0\}\);

(b) \(\exists r > 0, \forall x \in K \setminus K_r, \exists y \in K_r : F(y) - F(x) \not\in S\) where \(K_r = K \cap [-r, r] \neq \emptyset\).

**Proof:** (a) \(\Rightarrow\) (b): Suppose to the contrary that for all \(n \in \mathbb{N}\) there exists \(x_n \in K \setminus K_r, F(y) - F(x_n) \in S\) for all \(y \in K\). Then, for \(y \in K\) fixed and \(n > |y|\), we have \(F(y) - F(x_n) \in S\) for all \(n > |y|\). Assume that \(\frac{x_n}{|x_n|} \to v \in K^\circ\). We consider only the case \(v = 1\) (the other is entirely similar). Let \(\lambda > 0\).
and choice \( n \) satisfying \( \{ x_n \}_{n=1}^\infty \) > max \( \| y + \lambda \| \), \( y \in Y \). As \( y + \lambda \in (y, x_n) \) and \( F(y) - F(x_n) \in S \), we obtain \( F(y + \lambda) - F(y) \in -S \). Thus, \( 1 = v \in R_y \), a contradiction.

(b) \( \Rightarrow \) (a): Take any \( x \in K \), and assume that \( 0 \neq v \in R_y \). Then, for some \( \lambda > 0, x + \lambda v \in K \setminus K \). Thus, by (b), there is \( y \in K, y \in K \) such that \( F(y) - F(x + \lambda v) \in S \). On the other hand \( x + \lambda v = y + \lambda v \) for some \( \lambda > 0 \) by convexity of \( K \subset \mathbb{R} \). Therefore \( F(y + \lambda v) - F(y) \in -S \), contradicting the choice of \( v \in R_y \).

The implication (a) \( \Rightarrow \) (b) in the preceding theorem remains valid if \( K \) is a subset of a finite dimensional space under the additional assumption of \( P \)-lower semicontinuity on \( F \), as proven in [16].

By numerical aspects one might be interested in knowing, a priori, when the solution set is bounded since, like in scalar optimization, convergence of minimizing sequences depends heavily on that property. In this spirit, next theorem plays an important role. The assumption on the complementary of \( S \) in the next theorem is satisfied when

\[
S = Y \setminus -(P \setminus (P)), \quad S = Y \setminus \text{int } P,
\]

with \( P \) being a (not necessarily pointed) convex cone.

**Theorem 3.6** Let \( 0 \neq K \subset \mathbb{R} \) be a closed convex set and \( 0 \in S \subset Y \) be a cone satisfying \( (Y \setminus S) + (Y \setminus S) \subset Y \setminus S \). Assume that \( F : K \to Y \) is semistrictly \((S)\)-quasiconvex and Hypothesis (A) is satisfied. The following assertions are equivalent:

(a) \( R_y = \{0\} \);

(b) \( \exists r > 0, \forall x \in K \setminus K_r, \exists y \in K_r : F(y) - F(x) \not\in S \), where \( K_r = K \cap [-r, r] \neq \emptyset \);

(c) \( E_y \) is nonempty and bounded.

**Proof:** (a) \( \iff \) (b): It follows from the previous result.

(b) \( \Rightarrow \) (c): For every \( n \in \mathbb{N} \), set \( K_n = \{ x \in K : | x | \leq n \} \). We may suppose, without loss of generality \( K_n \neq \emptyset \) for all \( n \in \mathbb{N} \). Let us consider the problem

\[
\text{find } \bar{x} \in K_n : F(y) - F(\bar{x}) \in S \quad \forall y \in K_n.
\]

By Hypothesis (A), it admits a solution, say \( x_n \in K_n \) for all \( n \in \mathbb{N} \). We choose \( n \in \mathbb{N} \), such that \( n \leq r < n + 1 \). We claim that \( x_n \) is also a solution to problem (10). In fact, suppose there exists \( y \in K \) with \( | y | > n + 1 \) satisfying \( F(y) - F(\bar{x} + 1) \in S \). By assumption, there exists \( y_n \in K_n \subseteq K_n : F(y_n) - F(\bar{x} + 1) \not\in S \). Thus,

\[
F(y_n) - F(\bar{x} + 1) = F(y_n) - F(y) + F(y) - F(\bar{x} + 1) \in (Y \setminus S) + (Y \setminus S) \subset Y \setminus S,
\]
a contradiction.

(c) \( \Rightarrow \) (a): It is a consequence of Theorem 3.4.

**Example 3.7** The semistrict \((S)\)-quasiconvexity of \( F \) cannot be deleted in the preceding theorem. Take \( K = [0, +\infty) \) and \( F = (f_1, f_2) \) with

\[
f_1(x) = e^{-x}, \quad f_2(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ e^{-x+1} & \text{if } x \geq 1. \end{cases}
\]
Here $E = \{0\}$, $E_w = \{0\}$. We obtain $F(0) - F(2) = (1 - e^{-2}, -e^{-3}) \notin \mathbb{R}^2 \setminus \{0\}$ but $F(1) - F(2) = (e^{-1} - e^{-2}, 1 - e^{-3}) \in \text{int } \mathbb{R}^2$. Hence, $F$ is neither semistrictly $(\mathbb{R}^2 \setminus (\text{int } \mathbb{R}^2))$-quasiconvex at 2 nor semistrictly $(\mathbb{R}^2 \setminus (\text{int } \mathbb{R}^2))$-quasiconvex at 2. Here $(b)$ does not hold for either $S = \mathbb{R}^2 \setminus (\text{int } \mathbb{R}^2)$ or $S = \mathbb{R}^2 \setminus (\text{int } \mathbb{R}^2)$. The preceding theorem has several useful consequences which are expressed in the two following corollaries. The next result valid for quasiconvex real functions will be used subsequently. It is taken from [8], see also [20].

**Lemma 3.8.** If $f$ is quasiconvex on $\mathbb{R}$, then there is $a \in [-\infty, +\infty]$ such that $f$ is non-increasing on $(-\infty, a]$ and non-decreasing on the rest (with the convention $(-\infty, -\infty) = [+\infty, +\infty) = \emptyset$).

**Corollary 3.9.** Let $\emptyset \neq K \subseteq \mathbb{R}$ be convex and closed, $P \neq Y$ be a closed convex cone in $Y$, and $F : K \to Y$ be given.

(a) Set $D = \{p^* \in P : \|p^*\| = 1\}$. Assume that for all $p^* \in D$, $x \in K \mapsto \langle p^*, F(x) \rangle$ is lsc.

(a1) If $F$ is semistrictly $(Y \setminus (-P \setminus (P)))$-quasiconvex then,

$$E \neq \emptyset \text{ and bounded } \Rightarrow \text{argmin}_K \{q^*, F(\cdot)\} \neq \emptyset \forall q^* \in D.$$  

(a2) If for all $p^* \in D$, $x \in K \mapsto \langle p^*, F(x) \rangle$ is semistrictly quasiconvex, then,

$$E \neq \emptyset \text{ and } E \neq \mathbb{R} \Rightarrow \exists q^* \in D, \text{argmin}_K \{q^*, F(\cdot)\} \neq \emptyset.$$  

(b) Assume that $P^*$ is the weak-star closed convex hull of extre $P'$ (see the previous section), and that for all $p^* \in \text{extr} P'$, $x \in K \mapsto \langle p^*, F(x) \rangle$ is lsc.

(b1) If $F$ is semistrictly $(Y \setminus (-P \setminus (P)))$-quasiconvex then,

$$E \neq \emptyset \text{ and bounded } \Rightarrow \text{argmin}_K \{q^*, F(\cdot)\} \neq \emptyset \forall q^* \in \text{extr} P'.$$  

(b2) If for all $p^* \in \text{extr} P'$, $x \in K \mapsto \langle p^*, F(x) \rangle$ is semistrictly quasiconvex then,

$$E \neq \emptyset \text{ and } E \neq \mathbb{R} \Rightarrow \exists q^* \in \text{extr} P', \text{argmin}_K \{q^*, F(\cdot)\} \neq \emptyset.$$  

**Proof:** We only prove $(b)$ ($(a)$ is similar). We first observe that Hypothesis $(A)$ is satisfied: one can check that $\{x \in K : F(x) - F(y) \notin -P\}$ is closed for all $y \in K$ by virtue of (9). Thus, the result is a consequence of Theorem 3.6 since its Part $(b)$ reads

$$\exists r > 0, \forall x \in K \setminus K_r, \exists y \in K_r : F(y) - F(x) \notin -P \setminus l(P).$$  

This implies, in particular, that (see again (9))

$$\exists r > 0, \forall x \in K \setminus K_r, \exists y \in K_r, \forall q^* \in \text{extr} P', \langle q^*, F(y) \rangle \leq \langle q^*, F(x) \rangle.$$  

This inequality along with the lower semicontinuity of $\langle q^*, F(\cdot) \rangle$ imply that $\text{argmin}_K \{q^*, F(\cdot)\} \neq \emptyset$.

$(b)$ We know that lower semicontinuity and semistrict quasiconvexity imply quasiconvexity. Assume, to the contrary, that $\text{argmin}_K \{q^*, F(\cdot)\} = \emptyset$ for all $q^* \in \text{extr} P'$. Then, by the explicit qua-
siconvexity, either \( \langle q^*, F(i) \rangle \) is non-increasing on \( K \) for all \( q^* \in \text{extrd} \), or non-decreasing on \( K \) for all \( q^* \in \text{extrd} \). Let us consider \( K = [a, +\infty) \) (resp. \( K = (-\infty, a] \)) then for all \( q^* \in \text{extrd} \), \( \langle q^*, F(i) \rangle \) is strictly decreasing (resp. strictly increasing). We claim that \( E = \emptyset \). In fact, let \( \bar{x} \in K \) and take \( x > \bar{x}, x \in K \). Then

\[
\langle q^*, F(x) - F(\bar{x}) \rangle < 0 \quad \forall q^* \in \text{extrd}.
\]

It turns out that

\[
\langle q^*, F(x) - F(\bar{x}) \rangle \leq 0 \quad \forall q^* \in P^*.
\]

Both inequalities imply

\[
F(x) - F(\bar{x}) \in -P \setminus l(P),
\]

showing that \( \bar{x} \notin E \), proving the claim.

We consider the case \( K = \mathbb{R} \). Since \( E \neq \emptyset \), it is not difficult to prove the existence of \( p^*, q^* \in \text{extrd} \) such that \( \langle p^*, F(i) \rangle \) is strictly increasing and \( \langle q^*, F(i) \rangle \) is strictly decreasing. This allows us to prove that \( E = \mathbb{R} \), a contradiction.

We recall that if \( P \) is a closed convex cone with \( \text{int} P \neq \emptyset \), then \( P^* \) is the weak-star closed convex hull of \( \text{extrd} P^* \) by Remark 2.2 in [3].

**Corollary 3.10.** Let \( 0 \neq K \subseteq \mathbb{R} \) be convex and closed, \( P \neq Y \) be a closed convex cone in \( Y \) such that \( \text{int} P \neq \emptyset \), and \( F : K \rightarrow Y \) be given.

(a) If \( F \) is semistrictly \((Y \setminus \text{int} P)\)-quasiconvex and \( P \)-lsc, then

\[
E_w \neq \emptyset \quad \text{and bounded} \Rightarrow \arg\min_{p^*} \langle p^*, F(i) \rangle \neq \emptyset \quad \text{and compact} \quad \forall p^* \in D,
\]

where \( D = \{ p^* \in P^* : ||p^*|| = 1 \} \).

(b) Assume that \( F \) is \( P \)-lower semicontinuous on \( K \) and

(b1) \( \arg\min_{q} \langle q, F(i) \rangle \neq \emptyset \) for all \( q \in D \), with \( D \) as in (a). If for all \( q \in D \), the function \( \langle q, F(i) \rangle \) is semistrictly quasiconvex, then

\[
E_w = \overline{\text{co}} \left( \bigcup_{q \in D} \arg\min_{p^*} \langle q, F(i) \rangle \right).
\]

(b2) \( \arg\min_{q} \langle q, F(i) \rangle \neq \emptyset \) for all \( q \in \text{extrd} P^* \). If for all \( q \in \text{extrd} P^* \), the function \( \langle q, F(i) \rangle \) is semistrictly quasiconvex and \( P^* = \text{co}(\text{extrd} P^*) \) then

\[
E_w = \overline{\text{co}} \left( \bigcup_{q \in \text{extrd} P^*} \arg\min_{p^*} \langle q, F(i) \rangle \right).
\]

(c) Assume that \( \text{extrd} P^* \) is finite (i.e., \( P^* \) polyhedra) and

(c1) that for all \( p^* \in \text{extrd} P^* \),

\[
x \in K \mapsto \langle p^*, F(x) \rangle \text{ is quasiconvex and lsc. Then}
\]
E_w \neq \emptyset \text{ and } E_w \not\subset \mathbb{R} \Rightarrow \exists q^* \in \text{extrd}, \argmin_{x} \langle q^*, F(x) \rangle \neq \emptyset.

(c2) that for all \( p^* \in \text{extrd} \),

\( x \in K \mapsto \langle p^*, F(x) \rangle \) is semistrictly quasiconvex and lsc. Then

\[ E_w \neq \emptyset \text{ and compact } \iff \argmin_{x} \langle q^*, F(x) \rangle \neq \emptyset \text{ compact } \forall q^* \in \text{extrd} \quad P^* \text{.} \]

**Proof:** (a): Clearly Hypothesis (A) holds since \( P\)-lsc implies (actually equivalent to) the closedness of \( \{ x \in K : F(x) - F(y) \not\in \text{int } P \} \) for all \( y \in K \), see [4] (see also [14]). From (b) of Theorem 3.6 it follows that

\[ \exists r > 0, \forall x \in K \setminus K_r, \exists y \in K_r : F(y) - F(x) \in \text{int } P. \]

Taking into account (7), the previous relation becomes

\[ \exists r > 0, \forall x \in K \setminus K_r, \exists y \in K_r : \langle p^*, F(y) \rangle < \langle p^*, F(x) \rangle \forall p^* \in P^* \setminus \{0\}. \]

This inequality along with the (see Theorem 5.5 in [24, Chapter 1]) lower semicontinuity of \( \langle p^*, F(\cdot) \rangle \) implies that \( \argmin_{x} \langle p^*, F(\cdot) \rangle \neq \emptyset \) and compact for all \( p^* \in D \).

(b1) (b2) is similar: As in (a), for all \( q \in D, \langle q, F(\cdot) \rangle \) is lsc. Thus, \( \argmin_{x} \langle q, F(\cdot) \rangle \) is closed for all \( q \in D \). Since every lsc and semistrictly quasiconvex real-valued function is quasiconvex, we conclude that \( F \) is semistrictly \( (Y \setminus \text{int } P) \)-quasiconvex (see Proposition 2.8). By Theorem 3.4, \( E_w \) is convex. The closedness of \( E_w \) follows from the \( P \)-lower semicontinuity of \( F \), since it is equivalent to the closedness of \( \{ x \in K : F(x) - F(y) \not\in \text{int } P \} \) for all \( y \in K \) (see [4] or [14]), and \( E_w \) is the intersection of such sets. Therefore

\[ A_u \doteq \overline{\text{co}} \left( \bigcup_{q \in D} \argmin_{x} \langle q, F(\cdot) \rangle \right) \subseteq E_w. \]

Set \( h_y(x) = \langle q, F(\cdot) \rangle \). Thus, \( A_u \) is of the form \( (-\infty, +\infty), [\alpha, +\infty), (-\infty, \alpha), [\alpha, \beta] \) for some \( -\infty < \alpha \leq \beta < +\infty \). Obviously, in the first case there is nothing to prove. We only consider the case \( A_u = [\alpha, \infty) \). If \( x \in E_w \setminus A_u \) by (7) we may choose \( q \in D \) such that \( h_y(x) \leq h_y(\alpha) \). Take any \( x_y \in \argmin_{x} h_y \); then \( h_y(x_y) < h_y(x) \) since \( x \not\in A_u \), and therefore \( h_y(\alpha) < h_y(x) \) reaching a contradiction.

(c1) follows from a similar reasoning as in (b) of the previous corollary; (c2) is a consequence of (a) and (b2).

In the case when \( P^* \) is a polyhedral cone, that is,

\[ P^* = \bigcup_{i=0}^{t} \text{co} \{ a_1, \ldots, a_n \}, \]

it is known that (Corollary 5.6 [24, Chapter 1]) \( F \) is \( P \)-lsc if and only if \( \langle a_i, F(x) \rangle \) is lsc for all \( i = 1, \ldots, k \). In particular, Part (a) of Corollary 3.9 applies for instance when \( P = \mathbb{R}_+^{\ast}, F = (f_1, \ldots, f_n) \) with each \( f_i : K \to \mathbb{R} \) being lsc and semistrictly quasiconvex; and (a) of Corollary 3.10 can be applied if each \( f_i \) is lsc and quasiconvex.
The following examples show that our assumptions in Corollaries 3.9 and 3.10, are in some sense optimal.

**Example 3.11** (i) The function of Example 3.7 shows the necessity of the semistrict \((\mathbb{R}^2 \setminus \text{int} \mathbb{R}^2)\)-quasiconvexity or semistrict \((\mathbb{R}^2 \setminus \{0\})\)-quasiconvexity in (a) of the preceding two corollaries.

(ii) This example shows, in general, that (a) of Corollaries 3.9 and 3.10 fail to hold if \(E\) and \(E_w\) are unbounded. Consider \(P = \mathbb{R}^2_+\), \(f_1(x) = x\), \(f_2(x) = -x\), \(x \in \mathbb{R}\). Here \(E = E_w = \mathbb{R}\) and \(F = (f_1, f_2)\) is \(\mathbb{R}^2_+\)-convex.

(iii) In general, the reverse implication in (a) of Corollary 3.9 fails to hold. In fact, take \(K = [0, +\infty)\) and \(f_1(x) = 1\), \(f_2(x) = 2\). Here \(E = K = E_w\) and \(F = (f_1, f_2)\) is \(\mathbb{R}^2\)-convex.

(iv) In general, the reverse implication in (a) of Corollary 3.10 fails to hold. In fact, take \(K = [0, +\infty)\), \(P = \mathbb{R}^2_+\) and

\[
f_1(x) = \begin{cases} 1 & \text{if } x \in [1, 2], \\ 2 & \text{if } x \notin [1, 2]. \end{cases}
\]

\[
f_2(x) = \begin{cases} 1 & \text{if } x = 4, \\ 2 & \text{if } x \neq 4. \end{cases}
\]

Here \(D = \{(p_1', p_2') \in \mathbb{R}^2_+: \|\mathbf{p}\| = 1\}\). Set \(h_{p'}(x) = \langle p', F(x) \rangle\), \(p' \in D\), \(x \in K\).

After making some computations, we obtain

\[
h_{p'}(x) = \begin{cases} 2(p_1' + p_2'), & \text{if } x \in [0, 1]; \\ p_1' + 2p_2', & \text{if } x \in [1, 2]; \\ 2(p_1' + p_2'), & \text{if } x \in [2, 4]; \\ 2p_1' + p_2', & \text{if } x = 4, \\ 2(p_1' + p_2'), & \text{if } x \in [4, +\infty). \end{cases}
\]

and

\[
\arg\min_{p'} h_{p'} = \begin{cases} [1, 2], & \text{if } \min\{p_1' + 2p_2', 2p_1' + p_2'\} = p_1' + 2p_2', \\ 4, & \text{if } \min\{p_1' + 2p_2', 2p_1' + p_2'\} = 2p_1' + p_2'. \end{cases}
\]

Here \(E_w = [0, +\infty)\) and \(F = (f_1, f_2)\) is \(\mathbb{R}^2\)-quasiconvex and hence semistrictly \((\mathbb{R}^2 \setminus \text{int} \mathbb{R}^2)\)-quasiconvex.

(v) We now show the reverse implication in (b) of Corollary 3.9 and (c) of Corollary 3.10 may be false. Take the example in (ii) with \(K = \mathbb{R}\) to have \(E = \mathbb{R}\) and \(E_w = \mathbb{R}\); and consider \(K = [0, +\infty)\), \(f_1(x) = 1\), \(f_2(x) = e^{-x}\), to have \(E = 0\).

**Corollary 3.12** Let \(0 \neq K \subseteq \mathbb{R}\) be a closed convex set, \(P \subseteq Y\) be a convex cone. Assume that \(F : K \rightarrow Y\) is naturally \(P\)-quasiconvex, or equivalently, scalarly \(P\)-quasiconvex. Then, \(E_w \neq \emptyset\) and bounded \(\Rightarrow E \neq \emptyset\) and bounded.

**Proof:** From Proposition 2.6 and Theorem 3.6, it follows that \(R_e = \{0\}\), which implies that \(R_w = \{0\}\). It turns out that \(E\) is non-empty and bounded by the same theorem.
Example 3.13 (i) The boundedness on $E_{u}$ in the previous corollary cannot be deleted as the following example shows. Take $K = [0, +\infty)$, $f_{1}(x) = 1$, $f_{2}(x) = e^{-x}$; one obtains $E_{u} = [0, +\infty)$ whereas $E = \emptyset$. Here $F = (f_{1}, f_{2})$ is $\mathbb{R}_{+}^{2}$-convex.

(ii) The reverse implication does not hold as showed by taking $K = \mathbb{R}$. We consider $F : K \to \mathbb{R}^{2}$, $F = (f_{1}, f_{2})$ with $f_{1}(x) = |x|$, $f_{2}(x) = x$. Obviously $F$ is $\mathbb{R}_{+}^{2}$-convex (thus it is naturally $\mathbb{R}_{+}^{2}$-quasiconvex by Theorem 2.3). Here $E = \{0\}$ whereas $E_{u} = \mathbb{R}$.

In order to characterize the non-emptiness (possibly allowing unboundedness) of $E_{u}$, we consider the following non-coercive conditions:

(C1) for any sequence $\{x_{n}\}$ in $K$ satisfying:

(i) $|x_{n}| \to +\infty$, $|x_{n}| \to v \in R_{+}$, and,

(ii) for all $y \in K$ there exists $n_{y}$ such that $F(y) - F(x_{n}) \in S$ for all $n \geq n_{y}$, we assume the existence of $u \in K$ and $\bar{u}$, such that $|u| < |x_{n}|$ and $F(u) - F(x_{n}) \in -S$.

(C2) for every $x_{n} \in K$, $|x_{n}| \to +\infty$, there exists $\bar{u} \in \mathbb{N}$ and $u \in K$ such that $|u| < |x_{n}|$ and $F(u) - F(x_{n}) \in -S$.

(C3) there exists a nonempty compact set $D \subseteq K$ such that for all $x \in K \setminus D$ there exists $u \in D : F(u) - F(x) \in -S$.

(C4) there exist $u \in K$ and $r > |u|$ such that $F(u) - F(x) \in -S$ for all $x \in K$, $|x| = r$.

We point out that all of these conditions apply to situations in which the solution set may be unbounded. Notice that the cone $R_{n}$ is not explicitly mentioned in (C4), $i = 2, 3, 4, 5$. Clearly (C1) $\Rightarrow$ (C1), (C1) $\Rightarrow$ (C4) and (C1) $\Rightarrow$ (C4).

We are now in a position to establish various characterizations of the nonemptiness of $E_{u}$ when $K \subseteq \mathbb{R}$.

Theorem 3.14 Let $\emptyset \neq K \subseteq \mathbb{R}$ be a closed convex set, let $S \subseteq Y$ be a cone. Assume that $F : K \to Y$ is explicitly $(S)$-quasiconvex and Hypothesis (A) is satisfied. Then $E_{u}$ is convex and each of the conditions (C1) $\Rightarrow$ (C4) is equivalent to the non-emptiness of $E_{u}$.

Proof: The convexity of $E$ is a consequence of the remark above.

(C1) $\Rightarrow$ ($E_{u} \neq \emptyset$): For every $n \in \mathbb{N}$, set $K_{n} = \{x \in K : |x| \leq n\}$. We may suppose, without loss of generality $K_{n} \neq \emptyset$ for all $n \in \mathbb{N}$. Let us consider the problem (13). By Hypothesis (A), it admits a solution, say $x_{n} \in K_{n}$ for all $n \in \mathbb{N}$. If $|x_{n}| < n$ for some $n \in \mathbb{N}$, then, we claim that $x_{n}$ is also a solution of the problem (10). In fact, if there is $y \in K$ with $|y| > n$ such that $F(y) - F(x_{n}) \not\in S$. Take $z \in K$ with $z \in \text{co}(x_{n}, y)$ and $|z| < n$. By assumption, we have $F(z) - F(x_{n}) \not\in S$, which contradicts the choice of $x_{n}$. Therefore $x_{n}$ is a solution of the problem (10).

We consider now the case $|x_{n}| = n$ for all $n \in \mathbb{N}$. We may suppose, without loss of generality, that $x_{n} \to 0$. Clearly $v \in K^{+}$. We now check that $v \in R_{n}$. Take any $y \in K$ and $\hat{\lambda} > 0$. We consider the case $v = 1$ (when $v = -1$ the reasoning is similar). Then $y < y + \hat{\lambda}v = y + \hat{\lambda}$. We choose $x_{n}$ such that $x_{n} = n > \max \{\|y + \hat{\lambda}v\|, y\}$. Thus $F(y) - F(x_{n}) \in S$, which implies that $F(y + \hat{\lambda}v) - F(y)$...
\[ S \in -S \text{ since } y + \lambda \in (y, x_\lambda). \] Therefore 1 = \nu \in R_\nu. On the other hand, for any fixed \( y \in K \), \( F(y) - F(x_n) \in S \) for \( n \in \mathbb{N} \) sufficiently large (it is enough to take \( n > |y| \)). This shows that \( x_n \) satisfies (i) and (ii) of condition (\ast). Therefore there exists \( u \in K \) and \( \bar{\alpha} \), such that \(|u| < |x_n|\) and \( F(u) - F(x_n) \in -S \). We claim that \( x_n \) is also a solution to (10). If not, there is \( y \in K \), \( |y| > |u| - |x_n| |u| \) such that \( F(y) - F(x_n) \notin S \). On the other hand, carrying \( v \) in (i) or (ii), we have \( F(x_n) \in S \) for \( n \in \mathbb{N} \) sufficiently large (it is enough to take \( |y| > |x_n| |u| \)). This shows that \( x_n \) satisfies (i) and (ii) of condition (\ast). Therefore there exists \( u \in K \), \( \bar{\alpha} \), such that \( F(u) - F(x_n) \in S \), which contradicts the choice of \( x_n \). If \( x_n \in \text{co}(u, v)\) then \( F(y) - F(u) \in S \), since otherwise we obtain \( F(u) - F(x_n) \notin S \) contradicting the choice of \( y \). Thus, \( F(u) - F(y) \in S \). It follows that \( F(u) - F(y) \in S \), which contradicts the choice of \( y \). Therefore \( x_n \) is a solution to problem (10).

\[ (C_1) \Rightarrow (C_2); (C_2) \Rightarrow (C_3); (C_3) \Rightarrow (C_4); \] These are obvious.

\[ (E_2 \neq 0) \Rightarrow (C_2) : \text{Let } |x_n| \to +\infty \text{ and } u \in E_2. \] Thus, for \( \bar{\alpha} \) such that \(|x_n| \geq |u|\) we obtain \( F(x_n) - F(u) \in S \). This shows that \( F(u) - F(x_n) \in -S \).

\[ (E_3 \neq 0) \Rightarrow (C_3) : \text{Take any } u \in E_3 \text{ and set } D = \{u\}. \]

\[ (E_4 \neq 0) \Rightarrow (E_5) \Rightarrow (E_6) : \text{Take any } u \in E_5 \text{ and } r > |u|. \]

\[ (C_1) \Rightarrow (E_3 \neq 0) : \text{Let us consider problem (13) on } K, \text{ which admits a solution, say } x_. \text{ If } |x| < r \text{ we proceed as in the beginning of the proof (C_1) \Rightarrow (E_3 \neq 0) to conclude that } x_\lambda \in \text{co}(x, y) \text{ and } r > |u|. \]

\[ (C_4) \Rightarrow (E_5 \neq 0) : \text{If on the contrary } |x| = r \text{ there exists } u \in K, |u| < r \text{ such that } F(u) - F(x_n) \in -S. \] We reason as in the second part of the proof as above to deduce that \( x_\lambda \) is also solution to our original problem.

References