Indefinite Quadratic Bilevel Programming Problem with Multiple Objectives at Both Levels

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Abstract. Bilevel Programming has been proposed for dealing with decision process involving two decision makers with a hierarchical structure. They are characterized by the existence of two optimization problems in which the constraint region of the upper level problem is implicitly determined by the lower level optimization problem. In this paper, a general bilevel optimization problem with multiple objectives at both levels is considered. The objective functions at both levels are indefinite quadratic and the feasible region is assumed to be a convex polyhedron. An algorithm is developed to find an efficient solution of the bilevel programming problem.

Keywords: indefinite quadratic programming problem, Multi-objective programming, bilevel programming, efficient solution, quasi-concave function.

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Introduction

The bilevel programming structure is a class of hierarchical problem that show a two-stage decision making process when the constraint region of the first level problem is implicitly determined by another optimization problem.

General Bilevel Programming Problem (BLPP) is defined as,

\[(BLPP): \max_{X} f(X,Y)\]

where \(Y\) solves

\[\max_{Y} F(X,Y)\]

subject to \((X, Y)\in S,\)

where \(S = \{(X, Y) : AX + BY \leq b; X, Y \geq 0\}\).

Here, \(f(X, Y)\) and \(F(X, Y)\) can be linear or non-linear. (BLPP) has been used by researchers in several fields ranging from economics to transportation engineering. (BLPP) is also used to model problems involving multiple decision makers. These problems include traffic signal optimization [12] and genetic algorithms [4]. There are numerous methods to solve (BLPP). The most notable among them are cutting plane method [1,8,9], branch and bound methods [6,11,13] and the ranking method [2,10].

There are many planning and/or decision making situations that can be properly represented by a multi-objective programming model. Mathematically, a multi-objective programming problem (MOPP), is defined as,

\[(MOPP): \max_{X} \{f_1(X) = Z_1\} \max_{X} \{f_2(X) = Z_2\} \ldots \max_{X} \{f_k(X) = Z_k\}\]

subject to \(X\in S,\)

where \(S\) is a feasible set and \(f_i(X);\{j = 1,\ldots,k\},\) be linear/non-linear.

In the late seventies and early eighties, a lot of papers have been published on (MOPP) problems. White [14], in 1996, presented several equivalent mathematical programming formulation of the problem for maximizing a function over the efficient set, in case of a polytopal feasible region and linear function.

Indefinite Quadratic Programming Problem

Consider the indefinite quadratic programming problem (IQPP), given by
(IQPP): \[ \text{Max } Z(X) = Z_1(X)Z_2(X) = (C^T X + \alpha)(D^T X + \beta) \]
Subject to \[ AX \leq b \]
\[ X \geq 0, \]
where \( X \in \mathbb{R}^n; b \in \mathbb{R}^m; C, D \in \mathbb{R}^n; \alpha, \beta \in \mathbb{R} \) and \( A \in \mathbb{R}^{m \times n} \).

The feasible region \( S = \{ X : AX \leq b; X \geq 0 \} \) is non-empty and bounded.

Both \( Z_1(X) \) and \( Z_2(X) \) are positive for all \( X \in S \).

Thus, the function \( Z(X) \) is both quasi-concave and quasi-convex on \( S \). Therefore, the optimal solution to the problem (IQPP) occurs at an extreme point of \( S \).

Indefinite Quadratic Bilevel Programming Problem with Multiple Objective Functions at the Upper Level

The indefinite quadratic multi-objective bilevel programming problem (IQMBPP) can be formulated as,

\[ \text{(IQMBPP): Max} \left( f_i(X, Y), f_2(X, Y), \ldots, f_k(X, Y) \right) \]
subject to \[ A_1^i X + A_2^i Y \leq b_i \]
\[ X, Y \geq 0 \]
where for a given \( X, Y \) solves
\[ \text{Max } g(X, Y) \]
subject to \[ A_1^2 X + A_2^2 Y \leq b_2 \]
\[ Y \geq 0, \]
where \( f_i(X, Y) = Z_{1i}(X, Y)Z_{2i}(X, Y); \quad i = 1, \ldots, k \)
\[ Z_{1i}(X, Y) = c_{1i} X + d_{1i} Y + \alpha; \quad i = 1, \ldots, k \]
\[ Z_{2i}(X, Y) = c_{2i} X + d_{2i} Y + \beta; \quad i = 1, \ldots, k \]
\[ g(X, Y) = (p_1 X + p_2 Y + \gamma)(q_1 X + q_2 Y + \delta). \]

Here, \( X = \{x_1, \ldots, x_n\} \in \mathbb{R}^n, Y = \{y_1, \ldots, y_n\} \in \mathbb{R}^n \)
\[ c_{1i}, c_{2i} \in \mathbb{R}^n; \quad i = 1, \ldots, k; \quad d_{1i}, d_{2i} \in \mathbb{R}^n; \quad i = 1, \ldots, k \]
\[ \alpha, \beta \in \mathbb{R}; \quad i = 1, \ldots, k. \]
\[ A_1^i \in \mathbb{R}^{m \times n}; \quad A_2^i \in \mathbb{R}^{m \times n}; \quad b_i \in \mathbb{R}^m, \]
\[ A_1^2 \in \mathbb{R}^{m \times n}; \quad A_2^2 \in \mathbb{R}^{m \times n}; \quad b_2 \in \mathbb{R}^m, \]
\[ p_1, p_2 \in \mathbb{R}; \quad q_1, q_2 \in \mathbb{R}; \quad \gamma, \delta \in \mathbb{R} \]
The objective functions at both the levels are the product of two positive valued affine functions, hence, they are quasi-concave.

Let \[ S_0 = \{(X, Y) : A_1^T X + A_2^T Y \leq b^1; A_1^T X + A_2^T Y \leq b^2; X, Y \geq 0\} \]

The polyhedron \( S_0 \) defined by the constraint region of the (IQMBPP) problem is assumed to be non-empty and compact.

Define the lower level problem as follows:
For a given \( X = \bar{X} \geq 0 \), \( Y \) solves

\[
\text{(IQMBPP}(\bar{X})) : \begin{align*}
\text{Max } & g(X, Y) = (p_1 \bar{X} + p_2 Y + \gamma) (q_1 \bar{X} + q_2 Y + \delta) \\
\text{Subject to } & A_2^T Y \leq b^2 - A_1^T \bar{X} \\
& Y \geq 0,
\end{align*}
\]

The feasible region of the lower level problem is \( S(X) = \{Y : A_2^T Y \leq b^2 - A_1^T X; Y \geq 0\} \).

The inducible region or the feasible region of the upper level problem is defined as \( \bar{S} = \{(X, Y) \in S_0 : Y \text{ solves Max } g(X, Y), \text{ for a given } X\} \).

It is assumed that \( \bar{S} \) is non-empty and the optimal solution of (IQMBPP (\( \bar{X} \))) is singleton.

**Bilevel Feasible Solution**

Since \( \bar{S} \) is non-empty, therefore, for each value of the upper level problem, the lower problem has a feasible solution. Thus, any point of \( \bar{S} \) is a bilevel feasible solution.

**Efficient Solution**

A bilevel feasible solution \( (X, \bar{Y}) \in \bar{S} \) is an efficient solution of (IQMBPP) if there is no \( (X, Y) \in \bar{S} \), such that \( f_i(X, \bar{Y}) \leq f_i(X, Y) \), for \( i = 1, \cdots, k \) and \( f_j(X, \bar{Y}) < f_j(X, Y) \), for some \( j \in \{1, \cdots, k\} \).

**Efficient Set**

The set of all efficient solutions is denoted by \( (SE) \) and is called the efficient set.

**Solving (IQMBPP) Problem**

In order to solve (IQMBPP) problem, consider the multi-objective problem at the upper level, given by

\[
\text{(IQMULPP): } \begin{align*}
\text{Max} & (f_1(X, Y), f_2(X, Y), \cdots, f_k(X, Y)) \\
\text{subject to } & (X, Y) \in S_0.
\end{align*}
\]

Since \( S_0 \) is non-empty and compact and \( f_i's (i = 1, \cdots, k) \) are quasi-concave function, therefore, the optimal solution corresponding to each objective function \( f_i(X, Y), i = 1, \cdots, k \), lies at an extreme point of \( S_0 \). If this is also a point of \( \bar{S} \), then it is an extreme point of inducible region \( \bar{S} \), [3].
Consider (IQMULPP) problem with the first objective function \( f(X, Y) \). Let its optimal solution be \((X^1, Y^1)\). Because of our assumption that optimal solution is unique, it becomes an efficient solution of the given problem.

Otherwise, put \( X = X^i \) in the lower level problem. Let the optimal solution of this problem be \((X^i, Y^i)\).

If \( Y^i = Y_1 \), then \((X^i, Y^i)\) is a bilevel feasible solution. Find that \((X^1, Y^1)\) which is an efficient solution of (IQMBPP) problem.

If \( Y^1 \neq Y_1 \), then solve the problem (IQMULPP) with second objective function and repeat the process till an efficient solution of (IQMBPP) problem is obtained. If not, then find the second best solution of first objective function and repeat the process with the objective functions \( f_i(X, Y); i = 2, \ldots, k \). Finally, we will get an extreme point which is an efficient solution of (IQMBPP) problem because extreme points of \( S \) are contained in \( S_0 \).

**Algorithm to Solve (IQMBPP) Problem**

**Step 1**
Consider the (IQMULPP) problem as
\[
\max \left( \sum_{i=1}^{k} f_i(X, Y) \right)
\]
subject to \((X, Y) \in S_0\)

**Step 2**
Set \( i = 1 \).
Let \((X^i, Y^i)\) be an optimal solution of \( f_i(X, Y) \) for \( i = 1, \ldots, k \).

**Step 3**
Put \( X = X^i \) in the follower’s problem, as
\[
\max \ g(X^i, Y) = (p_1X^i + p_2Y + \gamma)(q_1X^i + q_2Y + \delta)
\]
subject to \( A_i^1Y \leq b_i - A_i^2X^i \)
\[
Y \geq 0.
\]
Let \((X^i, y)\) be its optimal solution.

**Step 4.1**
If \( Y^i = y \), then \((X^i, Y^i)\) is a bilevel feasible solution of (IQMBPP)

**4.2**
If \( Y^i \neq y \), set \( i = i + 1 \).
Go to step 2 and repeat the process.

**4.3**
If no bilevel feasible solution is obtained, find the next best solution of the first objective function. Go to step 3 and repeat the process with \( f_i(X, Y); i = 2, \ldots, k \).

**Step 5**
From the set of bilevel feasible solutions, find the efficient solutions of the problem (IQMBPP).

**Example 1:** Consider the following indefinite quadratic bilevel programming problem with multi-objective functions at the upper level:

\[(IQMBPP): \max_{x_i} f_i(x_1, x_2, x_3) = (x_i + 2x_1 + 3)(3x_2 + 2)\]
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Max \( f_2(x_1, x_2, x_3) = (2x_1 + x_2 + 2)(x_3 + 1) \)
subject to \( 3x_1 + x_2 + 2x_3 \leq 5 \)
\( x_2 + x_3 \leq 3 \)
where \((x_2, x_3)\) solves
Max \( g(x_1, x_2, x_3) = (x_2 + 1)(x_1 + x_2 + x_3 + 3) \)
subject to \( x_1 + 2x_2 + x_3 \leq 2 \)
\( 3x_2 + 2x_3 \leq 6 \)
\( x_1, x_2, x_3 \geq 0. \)

Solution: Consider the multi-objective problem at the upper level, given by

(IQMULPP): Max \( f_1(x_1, x_2, x_3) = (x_1 + 2x_3 + 3)(3x_2 + 2) \)
Max \( f_2(x_1, x_2, x_3) = (2x_1 + x_2 + 2)(x_3 + 1) \)
subject to \( 3x_1 + x_2 + 2x_3 \leq 5 \)
\( x_2 + x_3 \leq 3 \)
\( x_1 + 2x_2 + x_3 \leq 2 \)
\( 3x_2 + 2x_3 \leq 6 \)
\( x_1, x_2, x_3 \geq 0. \) \hspace{1cm} (1)

Solve the first objective

Max \( f_1(x_1, x_2, x_3) = (x_1 + 2x_3 + 3)(3x_2 + 2) \)
subject to the constraints (1).
The optimal table for the above problem is

| \( c_j \rightarrow \) | \( 1 \) | \( 0 \) | \( 2 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( d_j \rightarrow \) | \( 0 \) | \( 3 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |

The optimal solution of \( f_1(x_1, x_2, x_3) \) is \((0, 1, 0)\).
Put \( x_1 = 0 \) in the lower level problem, we get

Max \( g(0, x_2, x_3) = (x_2 + 1)(x_2 + x_3 + 3) \)
subject to $2x_2 + x_3 \leq 2$
$3x_2 + 2x_3 \leq 6$
$x_2, x_3 \geq 0.$

The optimal solution of the lower level problem is $\langle 1, 0 \rangle$.
Thus, $\langle 0, 1, 0 \rangle$ is a bilevel feasible solution.

Again, repeating the process with the second objective function, $f_2(x_1, x_2, x_3)$, of the upper level,
we get $x_1 = 1, x_2 = 0, x_3 = 1$.

Putting $x_1 = 1$ in the lower level problem, we find that $\langle 1, 0, 1 \rangle$ is not a bilevel feasible solution.

Finding the second best solution of $f_1(x_1, x_2, x_3)$, we find that $\left( \frac{8}{5}, \frac{1}{5}, 0 \right)$ is a bilevel feasible solution. Repeating the same method for $f_2(x_1, x_2, x_3)$, bilevel feasible solution is $\left( \frac{8}{5}, \frac{1}{5}, 0 \right)$.

Thus, the efficient bilevel feasible solutions for the above problem are $\langle 0, 1, 0 \rangle$ and $\left( \frac{8}{5}, \frac{1}{5}, 0 \right)$.

The efficient solutions for the problem (IQMBPP) are $\langle 0, 1, 0 \rangle$ and $\left( \frac{8}{5}, \frac{1}{5}, 0 \right)$.

### Indefinite Quadratic Bilevel Programming Problem with Multiple Objective Functions at Both Levels

In this section, we will be considering the indefinite quadratic bilevel program with multi-objective functions at both levels (IQBMPBL). It is defined as,

\[(IQBMPBL): \quad \begin{align*}
& \text{Max} \left( g_1(X,Y), g_2(X,Y), \cdots, g_n(X,Y) \right) \\
& \text{subject to } B_1^iX + B_2^iY \leq u^i \\
& \text{where } Y \text{ solves } \\
& \text{Max} \left( h_1(X,Y), h_2(X,Y), \cdots, h_n(X,Y) \right) \\
& \text{subject to } B_1^iX + B_2^iY \leq u^i \\n& \quad X, Y \geq 0,
\end{align*} \]

where $g_i(X,Y) = (p_{i1}X + q_{i1}Y + \gamma_i) (p_{i2}X + q_{i2}Y + \delta_i); i = 1, \cdots, s$;
$h_j(X,Y) = (a_{j1}X + b_{j1}Y + \eta_j) (a_{j2}X + b_{j2}Y + \varphi_j); j = 1, \cdots, t$;

Here, $X = \{x_1, \cdots, x_n\} \in \mathbb{R}^n$, $Y = \{y_1, \cdots, y_n\} \in \mathbb{R}^n$,
$p_{i1}, p_{i2} \in \mathbb{R}^n; i = 1, \cdots, s$
$q_{i1}, q_{i2} \in \mathbb{R}^n; i = 1, \cdots, s; \gamma_i, \delta_i \in \mathbb{R} \; : i = 1, \cdots, s$
$a_{j1}, a_{j2} \in \mathbb{R}^n; j = 1, \cdots, t$
$b_{j1}, b_{j2} \in \mathbb{R}^n; j = 1, \cdots, t; \eta_j, \varphi_j \in \mathbb{R}; j = 1, \cdots, t$.
\[ B_1^i \in \mathbb{R}^{m_i \times n}; \quad B_2^i \in \mathbb{R}^{m_i \times n}; \quad u^1 \in \mathbb{R}^m \]
\[ B_1^2 \in \mathbb{R}^{m_i \times n}; \quad B_2^2 \in \mathbb{R}^{m_i \times n}; \quad u^2 \in \mathbb{R}^m. \]

Let \( S_i = \{(X,Y) : B_1^i X + B_2^i Y \leq u^1; B_1^2 X + B_2^2 Y \leq u^2; X, Y \geq 0\} \).

Here, the constraint region \( S_i \) is a polyhedron for the problem (IQBMPBL). Also, \( S_i \) is assumed to be non-empty and compact.

The lower level problem can be defined as,

\[(IQBMPBL (X_i)): \max \{h_1(X,Y), h_2(X,Y), \ldots, h_t(X,Y)\} \]
subject to \( B_2^i Y \leq u^2 - B_1^i X, \quad Y \geq 0. \)

The feasible region of the lower level problem is \( T(X) = \{Y : B_2^i Y \leq u^2 - B_1^i X, \quad Y \geq 0\}. \)

The inducible region of the upper level problem is \( \bar{T} = \{(X,Y) \in S_i : Y \text{ solves } \max \{h_1(X,Y), h_2(X,Y), \ldots, h_t(X,Y)\}, \text{ for a given } X\}. \)

Again, \( \bar{T} \) is assumed to be non-empty and the optimal solution of \( (IQBMPBL (X_i)) \) is singleton.

**Algorithmic Development for (IQBMPBL) Problem**

Here, \( S_i \) is non-empty and compact set. The objective functions at both the levels, that is, \( g_i \)'s \((i = 1, \ldots, s)\) at the upper level and \( h_j \)'s \((j = 1, \ldots, t)\) at the lower level, are the product of two positive valued affine functions. Hence, the objective functions at both the levels are quasi-concave functions.

Consider the multi-objective problem at the upper level.

\[(IQBMPUL): \max \{g_1(X,Y), g_2(X,Y), \ldots, g_s(X,Y)\} \]
subject to \((X,Y) \in S_i. \)

Since \( g_i \)'s are quasi-concave functions, therefore, the optimal solution for each \( g_i (i = 1, \ldots, s) \) exists at an extreme point. If this is a point of \( \bar{T} \), then this is the extreme point of the inducible region, \( \bar{T} \), [3].

**Algorithm to Solve (IQBMPBL) Problem**

**Step 1**
Consider the \( (IQBMPUL) \) problem, defined as

\[(IQBMPUL): \max \{g_1(X,Y), g_2(X,Y), \ldots, g_s(X,Y)\} \]
subject to \((X,Y) \in S_i. \)

**Step 2**
Set \( i = 1. \)
Let \((X^*,Y^*)\) be an optimal solution of \( g_i(X,Y) \), for each \( i = 1, \ldots, s. \)
Step 3 Put $X = X^i$ in the follower’s problem, as
\[
\max \ h_j(X', Y), \quad j = 1, \ldots, t,
\]
subject to \[ B_j^i Y \leq u^i - B_j^i X^i \]
\[ Y \geq 0. \]

Step 4 Set $j = 1$.
Let $(X', Y)$ be an optimal solution of $h_j(X', Y)$.

Step 5.1 If $Y' = Y_j$, then $(X', Y')$ is a bilevel feasible solution of (IQBMPBL) problem.

Step 5.2 If $Y' \neq Y_j$, set $j = j + 1$. Go to step 4 and repeat the process.

Step 5.3 If $Y' \neq Y_j$, for any $j = 1, \ldots, t$; set $i = i + 1$. Go to step 2 and repeat the method.

Step 5.4 If no bilevel efficient feasible solution is obtained for any $i = 1, \ldots, s$ and for any $j = 1, \ldots, t$; find the next best solution of the first objective function, $g_i(X, Y)$. Go to Step 3 and repeat the procedure with $g_i(X, Y); i = 2, \ldots, s$.

Step 6 From the set of bilevel feasible solutions, find efficient solutions for the problem (IQBMPBL).

Example 2: Consider the following indefinite quadratic bilevel programming problem with multi-objective functions at both levels,

(IQBMPBL): \[
\begin{align*}
\max g_1(x_1, x_2, x_3, x_4) &= (2x_1 + x_2 + x_3 + 3)(x_2 + 2x_4 + 2) \\
\max g_2(x_1, x_2, x_3, x_4) &= (x_1 + 2x_3 + x_4 + 4)(x_2 + x_3 + 2) \\
\text{subject to} &\quad 2x_1 + x_2 + x_3 \leq 8 \\
&\quad 2x_2 + x_3 + x_4 \leq 6 \\
&\quad 2x_1 + x_2 + x_4 \leq 12 \\
\end{align*}
\]

where $(x_1, x_3)$ solves
\[
\begin{align*}
\max h_1(x_1, x_2, x_3, x_4) &= (2x_1 + x_3 + x_4 + 1)(x_2 + x_3 + x_4 + 3) \\
\max h_2(x_1, x_2, x_3, x_4) &= (2x_1 + x_4 + x_3 + 2)(x_3 + x_2 + 4) \\
\text{subject to} &\quad -2x_1 + 4x_2 + x_3 + 2x_4 \leq 16 \\
&\quad x_2 + 2x_3 + x_4 \leq 5 \\
&\quad x_1, x_2, x_3, x_4 \geq 0.
\end{align*}
\]

Solution: Using the algorithm, the bilevel feasible solutions of the above problem are,
\[
(7/2, 1, 0, 4), (7/2, 0, 0, 5), (0, 1, 0, 4), (0, 0, 0, 5), (11/4, 0, 5/2, 0), (13/6, 7/3, 4/3, 0), (0, 0, 5/2, 0), (25/6, 1, 0, 4), (0, 7/3, 4/3, 0).
\]
The set of efficient solutions for the problem (IQBMPBL) is
\[
\{(0, 0, 0, 5), (0, 0, 5/2, 0), (0, 7/3, 4/3, 0)\}.
\]
References