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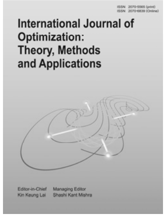

On the Behaviour of Functions around Zero-Derivative Points[♣]

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Abstract. Zero-derivative points are of the fundamental importance in calculus and optimization. It has been recently shown that on intervals around zero-derivative points, and only around zero-derivative points, every smooth function with a Lipschitz derivative is an "envelope" of a parabolloid. In this paper we give two equivalent, but geometrically different, reformulations of this result. They are applied to the three classic theorems: Fermat's extreme value theorem, the mean value theorem, and the Lagrange multiplier theorem. These theorems are augmented over intervals and stated without derivatives.

Keywords: zero-derivative point, Fermat's extreme value theorem, mean value theorem, Lagrange multiplier theorem.

AMS Classification: 26A06, 26A24, 26B35, 90C30, 90C46, 90C56

1 Introduction

The study of derivatives and their applications is an important basic topic in calculus, e.g., [1, 3, 4]. In particular, it is important to identify points where the derivative is equal to zero. According to Fermat's extreme value theorem, e.g., [4], only at these points functions can achieve extreme values. The text-books in calculus typically contain sections where extreme problems from various areas

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of sciences are formulated and solved after calculating the derivative and its zero-derivative points, e.g., [1, 3]. These problems may not always be easy to solve because calculation of the derivative $f'(x^*)$ at a given point x^*, and then checking whether $f'(x^*)=0$, may be time consuming and it may lead to errors after lengthy algebraic manipulations. Moreover, this standard approach is of local character and it does not describe the behaviour of the function on intervals around zero-derivative points.

It is desirable to have alternative criteria for identifying zero-derivative points without requiring calculation of the derivative, particularly if they describe the behaviour of functions on intervals around such points. This paper provides these criteria for smooth functions with a Lipschitz derivative. They are equivalent to the characterization that was recently given in [5] but have different geometric interpretations. In Section 2 we formulate the possibly new criteria for scalar functions and use them to describe the behaviour of functions around local extrema.

In Section 3 we apply the new criteria to the mean value theorem. This is possible because the points of average (mean) rate of change of the function f are the zero-derivative points of an auxiliary function that is used in the proof by Rolle's theorem, e.g., [1, 3, 4]. The existence of a point c where the instantaneous and the average rate of change coincide is guaranteed by the mean value theorem but this classic result does not characterize such points. Here we do characterize *c* for smooth functions with a Lipschitz derivative: For a given f and an arbitrary point c in the interior of an arbitrary interval $I=[a,b]$, we find conditions which are both necessary and sufficient that $f'(c)=[f(b)-f(a)]/(b-a)$. These characterizations describe *f* globally on I and they do not require the derivative of *f*.

In Section 4 we study functions in several variables and characterize their zero-derivative points. The results are used to describe the behaviour of the Lagrange function on arbitrary compact convex sets around constrained local extrema, e.g., [1, 5]. They suggest alternative approaches to the study of optimization problems with side constraints.

We study only smooth functions with a Lipschitz derivative. These are continuously differentiable functions with the property that their derivatives globally satisfy the Lipschitz condition. Loosely speaking, "almost all" smooth functions of practical interest, i.e., smooth functions used in modelling of real-life problems, satisfy this requirement. In particular, the requirement is satisfied for all twice differentiable functions. In mechanics, such functions describe the movements of objects that are governed by Newton's second law of motion.

2 Zero-Derivative Points for Functions of a Single Variable

The following classic result motivates this section. It is a necessary condition for a local extreme point.

Theorem 1. (*Fermat's Theorem*, [4, p.177].) If *f* has a local extremum (that is, maximum or minimum) at *c*, and if $f'(c)$ exists, then $f'(c)=0$.

We wish to characterize the behaviour of functions on intervals around a zero-derivative point c without using the derivative. In this section we study real functions of a single scalar variable f: $R \to R$. It is assumed that the functions are smooth on an open set containing an interval *I* = [*a*, *b*]. We also assume that the functions have a Lipschitz derivative on *I*. This means that there is a number $L \geq 0$ such that $|f'(x) - f'(y)| \leq L|x - y|$ for every *x* and *y* in *I*. Such *L* is called a Lipschitz constant of the derivative of f on *I*. Note that *L* is not uniquely determined and that it depends on *I*. If *L* is a Lipschitz constant, so is every $\tilde{L} \geq L$. Bigger the interval, by inclusion, the bigger *L*. If *f* is twice continuously differentiable, one can specify *L* to be the maximal absolute value of the second derivative f'' on I, i.e., $L = \max_{x \in I} |f''(x)|$.

The main result of this section is Theorem 2. It gives three results on zero-derivative points: one was recently given in [5] and the other two are possibly new. They are equivalent in the sense that any of the four statements implies the other three. Notation $I \setminus \{x^*\}$ stands for the set *I* without x^*, i.e., $I \setminus \{x^*\} = \{x : x \in I, x \neq x^*\}$.

Theorem 2. (*Characterizations of Zero-Derivative Points for Scalar Functions.*) Consider a smooth function $f : R \to R$ on an interval *I* = [*a*, *b*]. Assume that the derivative of *f* satisfies the Lipschitz condition on *I* with a constant *L* and consider an interior point x^* of *I*. Then the following statements are equivalent:

(i) $f'(x^*) = 0$.
(ii) $|f(x) - f(x^*)| \leq \Lambda(x - x^*)^2$ for some $\Lambda \geq 0$ and for every x in *I*, [5].
(iii) $1/\Lambda |f(x) - f(x^*)| \leq (x - x^*)^2$ for every $\Lambda > 0$ sufficiently large and for every x in *I*.
(iv) The ratio function $|f(x) - f(x^*)| / (x - x^*)^2$ is uniformly bounded on $I \setminus \{x^*\}$.

One can specify $\Lambda \geq \frac{1}{2}L$ in (ii) and (iii). Also $\Lambda \geq \frac{1}{2}L$ is a uniform bound in (iv).

The equivalence of (i) and (ii) was shown in [5], where (ii) was referred to as the "envelope characterization of zero-derivative points". Indeed, the function on the left-hand side of the inequality in (ii) is an "envelope" of the parabola appearing on the right-hand side. From this result one concludes that (ii) and (iii) and also (iii) and (iv) are equivalent. The statement (iii) is the "normalized envelope characterization of zero-derivative points". It says that all left-hand sides in the inequality, with "sufficiently large" Λ > 0, represent envelopes of the single "standard" parabola $P(x) = (x - x^*)^2$ with a vertex at x^*. The statement (iv) considers the ratio function of $f : \mathrm{R}(x, x^*) = |f(x) - f(x^*)| / (x - x^*)^2$ on $I \setminus \{x^*\}$. This is a symmetric function in two variables: x and x^*. The first step in solving extreme value problems by the standard method in calculus consists of differentiation and calculation of the roots of (i). In contrast, (iv) identifies points x^* for which the x-component of $\mathrm{R}(x, x^*)$ is uniformly bounded.

Example 1. In order to find the zero-derivative points of $f(x) = x^3$ we look along the $x^* -$ axis and consider the behaviour of $\mathrm{R}(\cdot, x^*)$ along the $x -$ axis. After division by $(x - x^*) \neq 0$, the ratio is $\mathrm{R}(x, x^*) = |(x^2 + xx^* + x^{*2}) / (x - x^*)|$. At $x^* = 0$, $\mathrm{R}(x, 0) = |x|$ is uniformly bounded on every finite interval while $\mathrm{R}(x, x^*) \to \infty$ as $x \to x^*$ for every $x^* \neq 0$. Hence we conclude that $x^* = 0$ is the only zero-derivative point. The behaviour of $\mathrm{R}(x, x^*)$ is depicted in Fig. 1 using histograms (bar charts), except at the origin, with the spacing of 0.25.

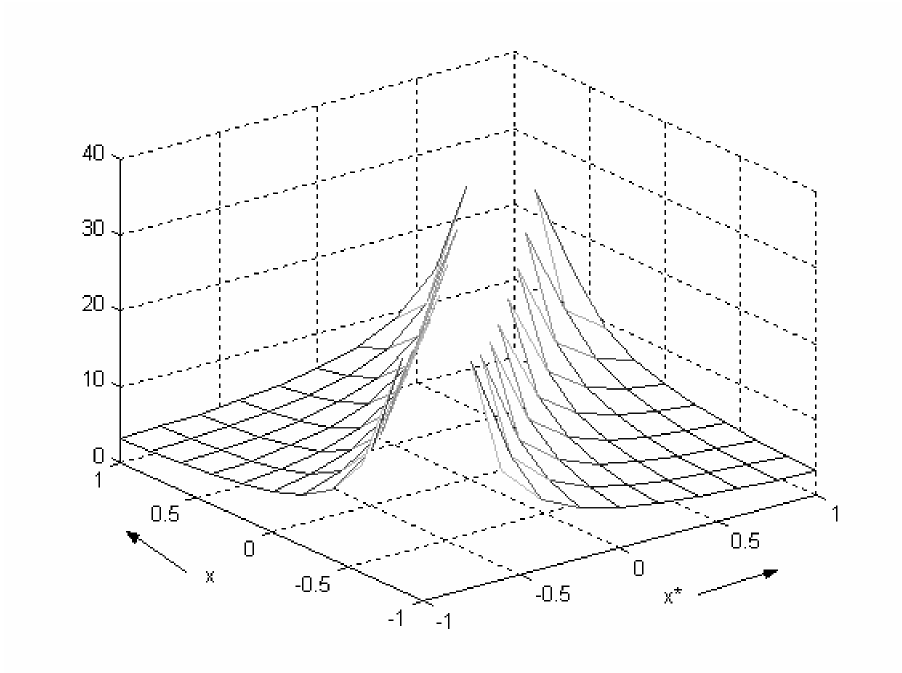

Fig. 1 Identification of a zero-derivative point

Example 2. We wish to know whether $f(x) = x^3 + x$ has a zero-derivative point. If x^* were such a point, then the ratio in (iv), i.e., $| x^3 + x + x^{*3} - x^* | /(x - x^*)^2$ would be uniformly bounded on some interval I with x^* in its interior. After division by $x - x^* \neq 0, | (x^2 + xx^* + x^{*2} + 1) / (x - x^*) |$ would be uniformly bounded. For this to happen, $x \to x^*$ would require that the numerator $3x^{*2} + 1 \to 0$, but this cannot happen. Hence we conclude that the given function does not have any zero-derivative point.

When there is no ambiguity, we use the term “criterion” instead of “characterization”. In the next two examples, we consider the global behaviour of specific functions on $I = [-1,1]$ to conclude whether or not the derivative is equal to zero at a given point x^* in the interior of *I*.

Example 3. Consider $f(x) = x^3 - x$. The ratio in (iv), after division by $x - x^* \neq 0$, is $| (x^2 + xx^* + x^{*2} - 1) / (x - x^*) |$. It is uniformly bounded if, and only if, the numerator converges to zero when $x \to x^*$, yielding $3x^{*2} = 1$. This says that there are two zero-derivative points: $x^* = \pm 1 / \sqrt{3}$.

Example 4. Let us look at the behaviour of $f(x) = x^3 - x$ around $x^* = +1 / \sqrt{3}$, using (iii). We have $1 / \Lambda | x^3 - x + 2\sqrt{3} / 9 | \leq (x - 1\sqrt{3})^2$ for every $\Lambda > 0$ sufficiently large, e.g., $\Lambda \geq \frac{1}{2} L$, and every x in *I*. Here the function on the left represents an envelope of the parabola on the right. The envelope condition is depicted in Fig. 2 for Λ = 2. A choice of larger Λ > 2 yields essentially the same figure.

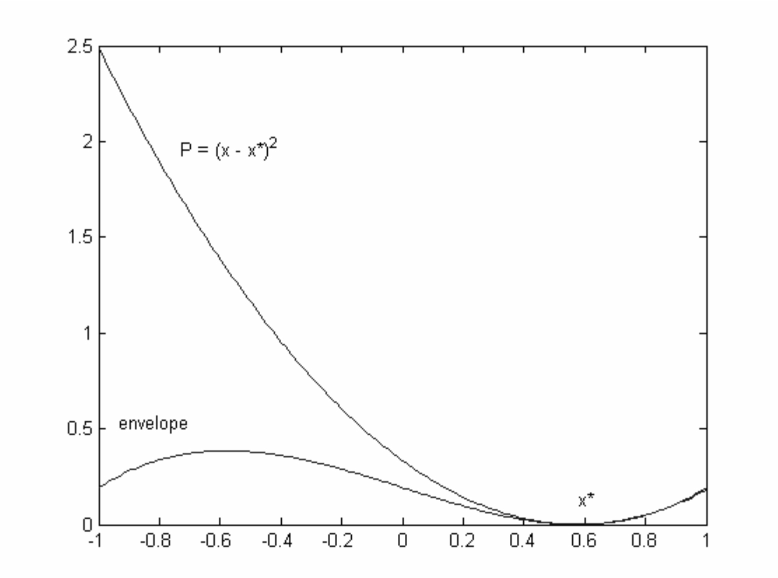

Fig. 2 Criterion (iii) at a zero-derivative point

In order to compare the behaviour of $f(x) = x^3 - x$ around zero- and non zero-derivative points, let us consider $x^* = 0$. Using, e.g., (iii), this point is a zero-derivative point if, and only if, $1/\Lambda\,|x^3 - x| \leq x^2$ for every $\Lambda > 0$ sufficiently large and for every $x \in I$. Violations of the inequality are depicted in Fig. 3 for $\Lambda = 2$ and $\Lambda = 4$. The functions on the left side of the inequality are not envelopes of $P = x^2$ regardless of $\Lambda > 0$. We conclude that $f'(0) \neq 0$.

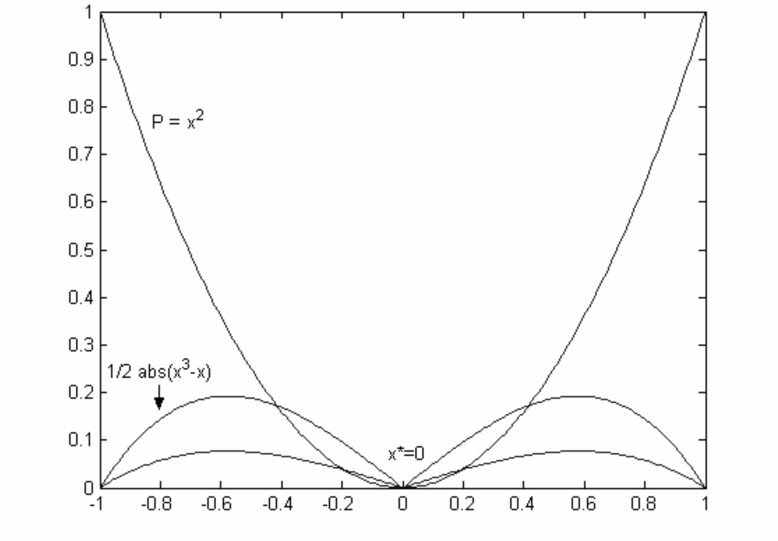

Fig. 3 Criterion (iii) at a non zero-derivative point

The behaviour of f around its zero-derivative point $x^* = +1/\sqrt{3}$, using the uniform boundedness criterion (iv) with the numerator $|x^3 - x + 2\sqrt{3}/9|$, the denominator $(x - 1/\sqrt{3})^2$, and their ratio $R(x, 1/\sqrt{3}) = x + 2/\sqrt{3}$ is depicted in Fig. 4. The behaviour around the non zero-derivative point $x^* = 0$ is depicted in Fig. 5.

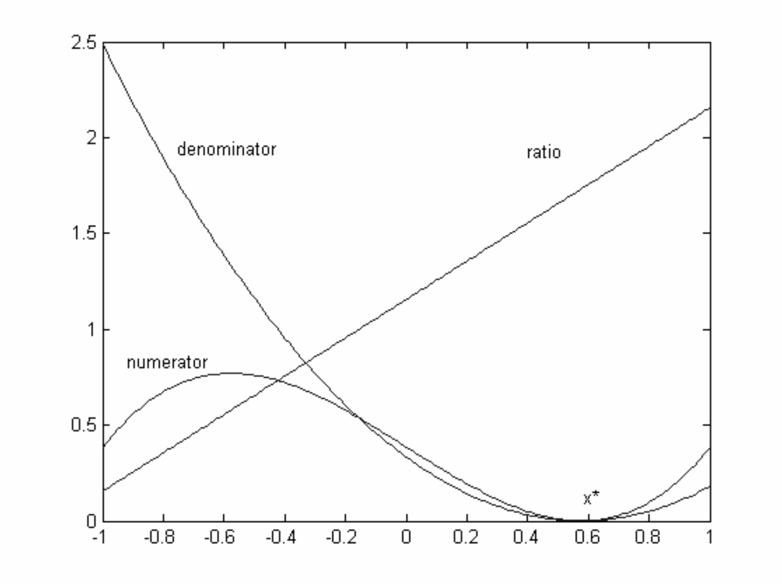

Fig. 4 Criterion (iv) at a zero-derivative point

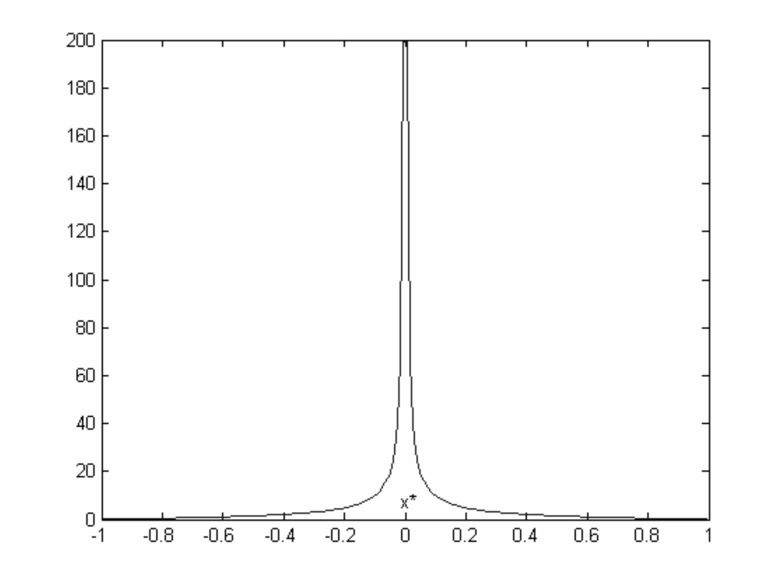

Fig. 5 Criterion (iv) at a non zero-derivative point

Using the new criteria, let us augment Fermat's Theorem 1 for smooth functions. Under slightly stronger assumptions, but without the derivative, we have the following.

Theorem 3. (*Behaviour of Functions Around Local Extrema.*) Consider a smooth function f on $I = [a, b]$ where it is assumed that the derivative of f has the Lipschitz property on I. If f has a local extremum at $x^* \in (a,b)$ then the statements (i), (ii), (iii) and (iv) of Theorem 2 hold.

Example 5. Consider $f(x) = x^3 \cos^2 x$ on $I = [-1,1]$. For $x^* = 0$, the envelope criterion (ii) is $|x\cos^2 x| \le \Lambda$ for some Λ and every $x \ne 0$. Since $|x\cos^2 x| = |x||\cos^2 x| \le |x| \cdot 1 \le 1$ on $I, x^* = 0$ can be a locally extreme point. At $x^* = \pi/4$, (iv) gives $|x^3 \cos^2 x|/(x-\pi/4)^2 \to \infty$ as $x \to \pi/4$. Hence $x^* = \pi/4$, can not be a locally extreme point. We have reached these conclusions without the derivative.

If f is smooth but its derivative does not have the Lipschitz property, then Theorems 2 and 3are not valid. Such situation is described by the following example:

Example 6. Function $f(x) = |x|^{3/2}$ is smooth but its derivative $f'(x) = 3/2\,|x|^{1/2}$ does not have the Lipschitz property around $x^* = 0$. The envelope criterion (ii) is here $|x|^{3/2} \le \Lambda x^2$ for some $\Lambda \ge 0$ and every x around $x^* = 0$. Such Λ does not exist, since $1/|x| \to \infty$, as $x \to x^*$, although $f'(x^*) = 0$.

3 The Mean Value Theorem

The mean value theorem was known to Parameshvara (1370-1460) from the Kerala school of astronomy and mathematics, e.g., [4]. The modern version given below was formulated by Lagrange. In celebration of the theorem, Beijing government has engraved the formula $f(b) - f(a) = f'(c)\,(b - a)$ on an overpass four blocks from the Tiananmen square. A photo of the formula is shown in [2]. Two other famous formulas: Newton's law $F = m_1 m_2 / r^2$ and Einstein's $E = mC^2$ are engraved next to the mean value theorem.

Theorem 4. (*Mean Value Theorem*, [1, p.217].) Let $f(x)$ be continuous on the closed interval $[a, b]$ and let $f'(x)$ exist for each x in the open interval (a, b). Then there exists at least one number $c \in (a, b)$ so that $f'(c) = [f(b) - f(a)]/(b - a)$.

We consider the following problem: If $f(x)$ is a smooth function with a Lipschitz derivative on $[a, b]$, what is a necessary and sufficient condition that, at an arbitrary given number $c \in (a, b)$, the instantaneous rate of change is equal to the average (mean) rate of change of f on I? In order to provide an answer let us consider the auxiliary function $d(x) = f(x) - f(a) - \{[f(b) - f(a)]/(b - a)\} \cdot (x - a)$. The mean value theorem is typically proved after applying Rolle's theorem to d, e.g., [1, 3, 4]. The key observation for us is that $d'(c) = 0$, at some point c if, and only if, $f'(c) = [f(b) - f(a)]/(b - a)$. Hence one can characterize c by applying Theorem 2 to d at its zero-derivative point c. In particular, the envelope characterization of optimality yields $|d(x) - d(c)| \le \Lambda\,(x - c)^2$ for some $\Lambda \ge 0$ and every x in I. After back substitution this is $|f(x) - f(c) - [f(b) - f(a)](x - c)/(b - a)| \le \Lambda(x - c)^2$ for some $\Lambda \ge 0$ and for every x in I. We also use the fact that a Lipschitz constant of $d'(x)$ is also a Lipschitz constant of $f'(x)$.

Theorem 5. (*Augmentation of the Mean Value Theorem.*) Consider a smooth function $f : R \to R$ on an interval $I = [a, b]$. Assume that the derivative of f satisfies the Lipschitz condition on I with a constant L and consider a number $c \in (a, b)$. Then the following statements are equivalent:

(i) $f'(c) = [f(b) - f(a)]/(b-a)$.

(ii) $|f(x) - f(c) - [f(b) - f(a)](x-c)/(b-a)| \leq \Lambda(x-c)^2$ for some $\Lambda \geq 0$ and for every x in I.

(iii) $1/\Lambda |f(x) - f(c) - [f(b) - f(a)](x-c)/(b-a)| \leq (x-c)^2$ for every $\Lambda > 0$ sufficiently large and for every x in I.

(iv) The ratio function $|f(x) - f(c) - [f(b) - f(a)](x-c)/(b-a)|/(x-c)^2$ is uniformly bounded on the set $I \setminus \{c\}$.

One can specify $\Lambda \geq 1/2L$ in (ii) and (iii). Also $\Lambda \geq 1/2L$ is a uniform bound in (iv).

A physical interpretation of d: If $f(x)$ describes the trajectory of an object moving from $(a, f(a))$ to $(b, f(b))$ in a time interval $[a, b]$, $a < b$, and if $\sec(x) = f(a) + \{[f(b) - f(a)]/(b-a)\} \cdot (x-a)$ describes the trajectory between the same points when the object is moving along a straight line (secant) at the average velocity $[f(b) - f(a)]/(b - a)$, then $d(x) = f(x) - \sec(x)$. Since at every x, one should subtract $d(x)$ from $f(x)$ to obtain the average rate-of-change value sec(x), we call $d(x)$ the "deviation function" (or the "vertical distance function", e.g., [1]).

Example 7. Consider the function $f(x) = x^3 - x$ on $I = [0, 2]$. Here $\sec(x) = 3x$, the deviation function is $d(x) = x^3 - 4x$, and $c = 2\sqrt{3}/3$ is both the instantaneous and the average rate of change point for f on I. Graphs of these objects are depicted in Fig. 6. Note that $d'(c) = 0$.

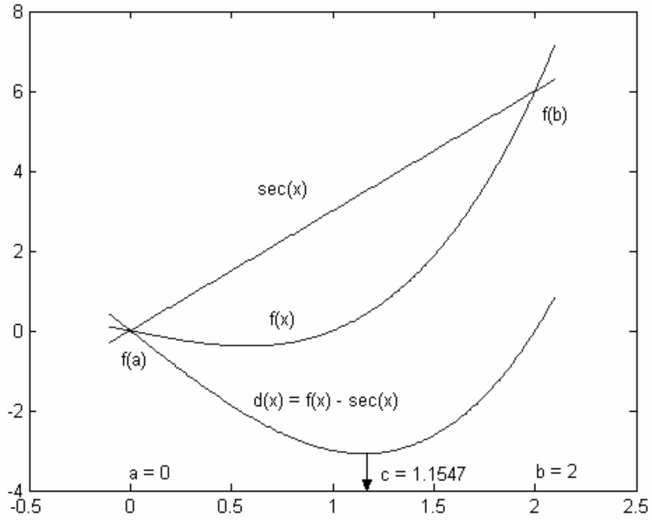

Fig. 6 Instantaneous and average rate of change point c

Example 8. Consider the function $f(x) = x^3 - x$. We wish to know whether $c = 1$ is an average rate of change point for f on I. The ratio in (iv), after division by $x - 1 \neq 0$, becomes $|x(x + 1) - 4| / |x - 1| \rightarrow \infty$ as $x \rightarrow 1$. We conclude that $c = 1$ is not a point of the average rate of change.

Theorem 4 claims that there is a number $c \in (a, b)$ which satisfies (i). The same c also satisfies (ii), (iii) and (iv), by Theorem 5. The four claims can be interpreted in words:

(i) The instantaneous rate of change at c is equal to the average rate of change of f on I.
(ii) The deviation function $d(x)$ has the quadratic envelope property around c and only around c on I.
(iii) The deviation function $d(x)$ has the normalized quadratic envelope property around c and only around c on I.
(iv) The ratio function of $d(x)$ is uniformly bounded around c and only around c on $I \setminus \{c\}$.

Example 8. Consider the function $f(x) = x^3 - x$ on $I = [0, 2]$. The deviation function $d(x) = x^3 - 4x$ has the normalized quadratic envelope property around the average rate of change point $c = 2\sqrt{3}/3$ of f on I, by property (iii). This situation is depicted in Fig. 7 with $\Lambda = 6$.

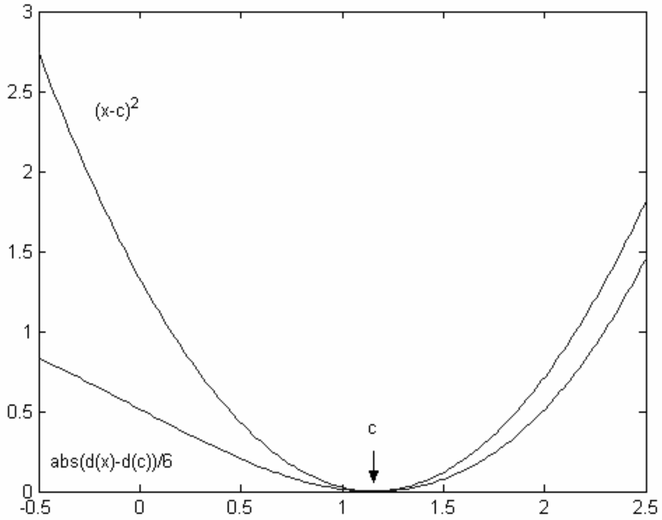

Fig. 7 Augmentation (iii) of the mean value theorem

4 Zero-Derivative Points for Functions in Several Variables

In this section we formulate Theorem 2 for functions in several variables. After minor adjustments, the results are used to describe the behaviour of the Lagrange function around local extrema subject to equality constraints.

Consider a smooth function in several variables $f:\mathrm{R}^n \to \mathrm{R}$ on a compact convex set K of its open domain in the Euclidean space R^n. Suppose that the (Fréchet) derivative of f at x, which is represented by the row gradient $\nabla f(x)$, has the Lipschitz property on K in R^n. In the n-dimensional setting this means that there exists a constant L, called a Lipschitz constant of the derivative on K, such that

$$\| \nabla^T f(x) - \nabla^T f(y) \| \le L \| x - y \|$$

for every x and y in K. Here $\nabla^T f(x)$ denotes the transposed $\nabla f(x)$ and the vector norm is chosen to be Euclidean: $\| u \| = (u^T u)^{1/2}$. We assume that K has interior points. For a given point x^* in the interior of K we wish to know whether or not $\nabla f(x^*) = 0$.

Theorem 6. (*Characterizations of Zero-Derivative Points for Functions in Several Variables.*) Consider a smooth function $f:\mathrm{R}^n \to \mathrm{R}$ on a compact convex set K with interior points. Assume that the derivative of f satisfies the Lipschitz condition on K with a constant L. Consider an interior point x^* of K. Then the following statements are equivalent:

(i) $\nabla f(x^*) = 0$.
(ii) $| f(x) - f(x^*) | \le \Lambda \| x - x^* \|^2$ for some $\Lambda \ge 0$ and for every x in K, [5].
(iii) $1/\Lambda \, | f(x) - f(x^*) | \le \| x - x^* \|^2$ for every $\Lambda > 0$ sufficiently large and for every x in I.
(iv) The ratio function $| f(x) - f(x^*) | / \| x - x^* \|^2$ is uniformly bounded on $K \setminus \{x^*\}$.

One can specify $\Lambda = ½ L$ in (ii) and (iii). This Λ is a uniform bound for (iv).

Example 9. Consider $f(x) = x_1^3 - x_2^3$ around the zero-derivative point $x^* = 0$ in the set $K = \{-2 \le x_1, x_2 \le 2\}$. Criterion (iii), with $\Lambda = 10$, is depicted in Fig. 8. Criterion (iv) is depicted in Fig. 9.

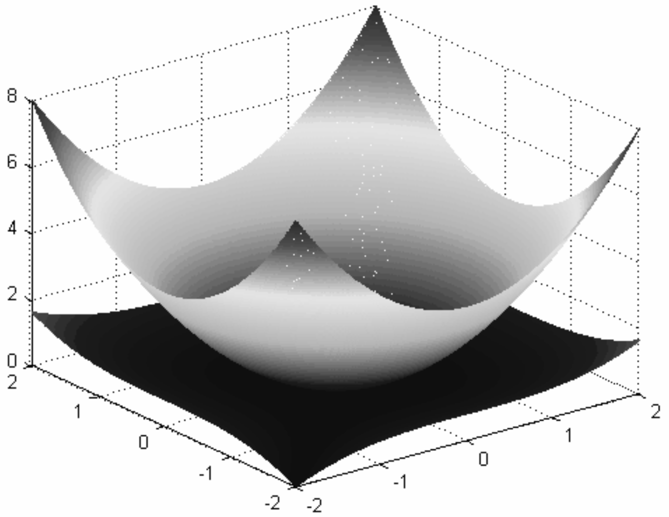

Fig. 8 Criterion (iii) at a zero-derivative point

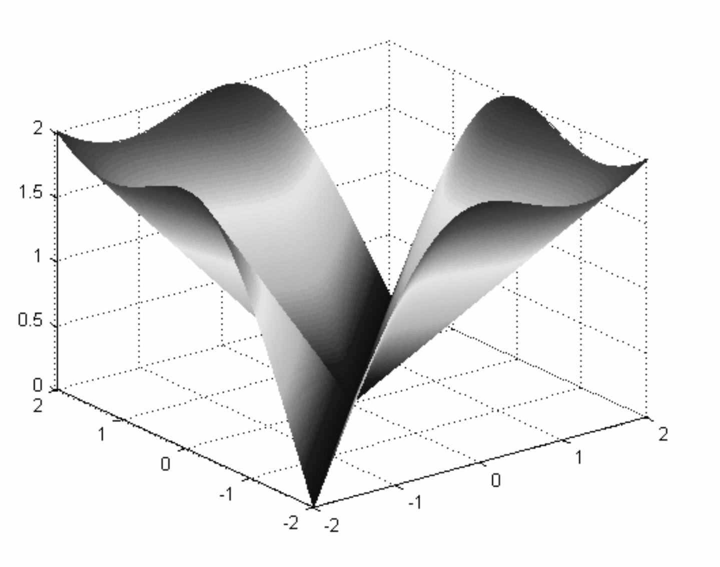

Fig. 9 Uniform bound criterion (iv) around a zero-derivative point

Theorem 6. can be used to augment the theorem of Lagrange for the problem of optimizing a function f^0 of n variables on the set of points (vectors) that is implicitly determined by a system of m equations in n variables:

$$\text{Opt } f^0(x), \text{ subject to } f^i(x)=0, i \in P=\{1,\cdots,m\}. \tag{1}$$

For this problem we construct the Lagrange function

$$L(x,\lambda)=\lambda_0 f^0(x)+\sum\nolimits_{i\in P}\lambda_i f^i(x)$$

where $\lambda=(\lambda_0,\lambda_1,\cdots,\lambda_m)$ is some non-zero (m +1)-tuple. The set of all points $x \in \mathrm{R}^n$, that satisfy the m constraints, is the feasible set of the problem and it is denoted by F. If $N(x^*)$ denotes a neighbourhood of x^*, then x^* is a constrained local minimum if $f^0(x^*) \le f^0(x)$ for every $x \in F \cap N(x^*)$. If $f^0(x^*) \ge f^0(x)$ for every $x \in F \cap N(x^*)$, then x^* is a constrained local maximum. If x^* is a constrained local minimum or maximum then x^* is a constrained local extremum. The classic Lagrange multiplier theorem says that, at such a point, the derivative of the Lagrange function is equal to zero: $\nabla L(x^*, \lambda) = 0$ for some non-zero (m + 1)-tuple λ. Theorem 6 provides three more equivalent conditions for a constrained local optimum:

Theorem 7. (*Behaviour of the Lagrange Function around Local Extrema.*) Suppose that a feasible x^* is a constrained local extremum of (1), where x^* is also an interior point of some arbitrary compact convex set K on which f^0, f^i, $i \in P$ are smooth functions with Lipschitz derivatives. Then there exists a non-zero $\lambda=(\lambda_0,\lambda_1,\cdots,\lambda_m)$ such that

(i) $\nabla L(x^*, \lambda) = 0$.

(ii) $|\lambda_0[f^0(x^*)-f^0(x)]-\sum\nolimits_{i\in P}\lambda_i f^i(x)| \le \Lambda \| x^*-x\|^2$,[5]
for some $\Lambda \ge 0$ and every x in K.

(iii) $1/\Lambda\,|\,\lambda_0[f^0(x^*)-f^0(x)]-\sum_{i\in P}\lambda_i f^i(x)\,|\le\|\,x^*-x\,\|^2$
for every $\Lambda > 0$ sufficiently large and for every x in I .

(iv) $|\,\lambda_0[f^0(x^*)-f^0(x)]-\sum_{i\in P}\lambda_i f^i(x)\,|/\|\,x-x^*\,\|^2$ is uniformly bounded on $K\setminus\{x^*\}$.

If a "regularization condition" is satisfied, which happens, e.g., if the gradients of all constraints are linearly independent at x^*, then one can specify $\lambda_0 = 1$ in $L(x, \lambda)$. (This is the form of the Lagrangian often used in calculus.) The process of solving the requirement $x^* \in F$, together with the statement (i) in Theorem 7 for x^* and λ, is called "The Method of Lagrange". Since the four statements are equivalent, one could, in principle, find candidates for constrained optima using any of them, not only (i). These might be directions for possible future research.

Example 10. Consider Opt x_1x_2, subject to $x_1 + x_2 = 10$. Here one can use the Lagrangian $L(x, \lambda) = x_1x_2 + \lambda(x_1 + x_2 - 10)$. The criterion (iv) says that

$$|\,x_1x_2+\lambda^*(x_1+x_2-10)-x_1^*x_2^*\,|/[(x_1-x_1^*)^2+(x_2-x_2^*)^2] \tag{2}$$

is uniformly bounded on a compact convex set containing local extrema in its interior. Here a local extremum satisfies $x_1^* + x_2^* = 10$ and λ^* is some multiplier. Since (2) is uniformly bounded also along every coordinate axis, one can specify $x_2 = x_2^*$ to simplify the ratio. After division by $|\,x_1 - x_1^*| \neq 0$ the ratio becomes

$$|10+\lambda^*-x_1^*\,|/|\,x_1-x_1^*\,|.$$

The numerator must be zero, yielding $\lambda^* = x_1^* - 10$. Similarly, after specification $x_1 = x_1^*$ in (2), we have $\lambda^* = x_2^* - 10$. Hence $\lambda^* = -5$, $x_1^* = x_2^* = 5$. We have found a candidate for a local optimum using (iv) instead of using (i) and the derivatives.

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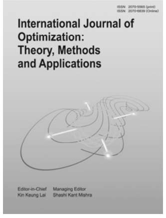

A Vector Optimization Approach to Cournot Oligopolistic Market Models

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Abstract. We propose a vector optimization approach to linear Cournot oligopolistic market equilibrium models. We use the scalarization technique to find a Pareto efficient solution to the model. It leads to a nonconvex quadratic program. We employ specific properties of this quadratic program to develop a relaxation algorithm for finding a Pareto efficient solution to the model. The total profit of the model at a Pareto efficient point, in general, is greater than that at the Nash equilibrium point. Some numerical experiments and results on randomly generated data are reported.

Keywords: Cournot oligopolistic market model, vector optimization approach, nonconvex quadratic, relaxation algorithm.

1 Introduction

The Cournot oligopolistic market model is one of fundamental models in economics. In this model it is assumed that there are n-firms producing a common homogeneous commodity. Each firm has a strategy set and a profit function. Actually, each firm seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other firms are parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept. In the linear Cournot model the profit function of firm i is given by

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$$f_i(x) = (\alpha - \beta\sum_{i=1}^{n} x_i)x_i - h_i(x_i)(i = 1,\cdots,n).$$

where $\alpha, \beta > 0$ and the cost function h_i is an affine function that depends only on the quantity x_i of firm *i*. In this linear case, it has been shown that (see e.g. [4]) the model has a unique Nash equilibrium point which is the unique solution of a strongly convex quadratic program. In the case h_i is not an affine function, the problem of finding a Nash equilibrium point can be formulated as a variational inequality (see [1,8] for h_i is convex and [6] for h_i is concave).

In this paper we propose a vector optimization approach rather than the equilibrium approach to linear Cournot market model. Namely, we are interested in Pareto efficient solutions of the vector function $f(x) := (f_1(x),\cdots, f_n(x))$ on the strategy set of the model. Our proposal is motivated by the fact that the total profit $\sum_{i=1}^{n} f_i(x)$ when x is a Pareto efficient point, in general, greater than that when x is the Nash equilibrium point. We use the scalarization technique to obtain a Pareto efficient point. The scalarization leads to a nonconvex quadratic program. Thanks to specific properties of this quadratic program we can develop a relaxation algorithm to obtain its a global optimal solution which is also a Pareto efficient point to the model. Numerical results on randomly generated data show that the proposed algorithm is efficient.

2 Preliminaries on the Linear Nash-Cournot Model

In the oligopolistic market model that we are going to consider, it is assumed that there are *n*-firms producing a common homogeneous commodity. The commodity's quantity of firm *i* is signed by $x_i (i = 1,\cdots,n)$, and $x^T = (x_1,\cdots,x_n)$ is the vector commodity's quantity of all *n* firms. In this paper, we assume that the price p_i of firm *i* depends on the total quantity $\sigma = \sum_{i=1}^{n} x_i$ of the commodity, and the cost function h_i of firm *i* depends only on the quantity $x_i (i = 1,\cdots,n)$.

The price function of firm *i* is defined by:

$$p_i(x) := \alpha_i - \beta_i\sum_{i=1}^{n} x_i,$$

where $\beta_i (i = 1,\cdots,n)$ is the reducing price coefficient of firm *i*. Naturally, the profit function of firm *i* has form:

$$f_i(x) = (\alpha_i - \beta_i\sum_{i=1}^{n} x_i)x_i - h_i(x_i).$$

Let $U_i \subset \mathbb{R}, (i = 1,\cdots,n)$ denote the stratery set of the firm *i*. In this paper, we suppose that:

$$U_i := \begin{cases} x_i \in R : 0 \leq x_i \leq b_i > 0(i = 1,\cdots, p \leq n); \\ x_i \in R : 0 \leq x_i (i = p+1,\cdots,n), \end{cases}$$

where $b_i (i = 1,\cdots, p)$ are scalar in *R*. A commonly used approach to this model is based upon the Nash equilibrium concept. This concept is used when each firm seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other firms are parametric input. In this context, a Nash equilibrium is a production pattern in which no firm can increase its profit by changing its controlled variables. Thus under this equilibrium concept, each firm determines its best response given other firms' actions. Mathematically, a point $x^* = (x_1^*, \cdots, x_n^*) \in U := U_1 \times \cdots \times U_n$ is said to be a Nash-equilibrium if

$$f_i(x_1^*, \cdots, x_{i-1}^*, y_i, x_{i+1}^*, \cdots, x_n^*) \leq f_i(x_1^*, \cdots, x_n^*) \forall y_i \in U_i, \forall i = 1, \cdots, n. \tag{2.1}$$

When $h_i(.)$ and $p(.)$ are affine, the market problem can be fomulated as a special Nash equilibrium problem in the n-person noncooperative game theory, which in turn is a strongly monotone variational inequality.

Let

$$\Psi(x, y) := -\sum_{i=1}^{n} f_i(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) \tag{2.2}$$

and

$$\Phi(x, y) := \Psi(x, y) - \Psi(x, x), \tag{2.3}$$

then it has been proved that the problem of finding an equilibrium point of this model can be formulated as the following equilibrium problem in the sense of Blum and Oettli.

$$\begin{cases} \text{find } x^* \in U \text{ such that} \\ \Phi(x^*, y) \geq 0 \ \forall \ y \in U. \end{cases} \tag{EP}$$

In classical Cournot models, the price and the cost functions for each firm are assumed to be affine of the forms

$$\begin{gathered} p_i(\sigma) \equiv p(\sigma) = \alpha_0 - \beta\sigma, \alpha_0 \geq 0, \beta > 0, \text{with} \sigma = \sum\nolimits_{i=1}^{n} x_i, \\ h_i(x_i) = \mu_i x_i + \xi_i, \mu_i \geq 0, \xi_i \geq 0 (i = 1, \cdots, n). \end{gathered} \tag{2.4}$$

In this case, using (2.1), (2.2) and (2.3). It is easy to check that

$$\Phi(x, y) = \left\langle \tilde{A}x + \mu - \alpha, y - x \right\rangle + y^T A y - x^T A x$$

where

$$A = \begin{pmatrix} \beta & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix} \tilde{A} = \begin{pmatrix} 0 & \beta & \beta & \cdots & \beta \\ \beta & 0 & \beta & \cdots & \beta \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta & \beta & \beta & \cdots & 0 \end{pmatrix}$$

and

$$\alpha^T = (\alpha_0, \cdots, \alpha_0), \mu^T = (\mu_1, \cdots, \mu_n).$$

Then Problem (EP) can be rewritten as the following mixed variational inequality

$$\left\langle \tilde{A}x+\mu-\alpha, y-x\right\rangle + y^T Ay - x^T Ax \geq 0 \ \forall\ y\in U.$$

Let

$$Q:=\begin{pmatrix} 2\beta & \beta & \beta & \cdots & \beta \\ \beta & 2\beta & \beta & \cdots & \beta \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta & \beta & \beta & \cdots & 2\beta \end{pmatrix}.$$

Clearly, since $\beta>0, Q$ is a symmetric and positive definite matrix. Since matrices $\tilde{A}$ and A are symmetric and $2A+\tilde{A}=Q$, this variational inequality can be reformulated equivalently as the following strongly convex quadratic programming problem:

$$\min_{x\in U}\{\frac{1}{2}x^T Qx+(\mu-\alpha)^T x\}. \tag{QP}$$

Hence this problem has a unique optimal solution which is also the unique equilibrium point of the classical oligopolistic market equilibrium model.

Observe that any solution of (QP) solves the problem

$$\max_{x\in U}\{\alpha^T x-\mu^T x-\frac{1}{2}x^T Qx\}. \tag{$QP1$}$$

This means that a Nash-equilibrium point is the point that maximizes total profit for all firms with reducing price coefficient matrix

$$\frac{1}{2}Q=\begin{pmatrix} \beta & \frac{\beta}{2} & \frac{\beta}{2} & \cdots & \frac{\beta}{2} \\ \frac{\beta}{2} & \beta & \frac{\beta}{2} & \cdots & \frac{\beta}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\beta}{2} & \frac{\beta}{2} & \frac{\beta}{2} & \cdots & \beta \end{pmatrix}.$$

On the other hand, by (2.1) and (2.4), without using equilibrium notion, as usual, the total profit for all firms is defined as

$$f(x):=\sum_{i=1}^{n} f_i(x)=\alpha^T x-\mu^T x-x^T Gx.$$

Thus to determine the production level that maximizes the total profits of all producers, we can solve the following quadratic programming problem

$$\max_{x\in U}\{\alpha^T x-\mu^T x-x^T Gx\} \tag{$QP2$}$$

where now the reducing price coefficient matrix is given by

$$G = \begin{pmatrix} \beta & \beta & \beta & \cdots & \beta \\ \beta & \beta & \beta & \cdots & \beta \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta & \beta & \beta & \cdots & \beta \end{pmatrix}.$$

So, for this affine case, the reducing price coefficient matrices of the equilibrium model and the optimization model are somewhat difference, but the both models lead to strongly convex quadratic optimization problems. Namely, the equilibrium model leads to (QP1) and the optimization model leads to (QP2).

In general, the reducing price coefficient of all firms is not the same and the price function $p(\sum_{j=1}^{n} x_j)$ can change from firm by firm. Namely, the price has the following form:

$$p_i(\sigma) := p_i(\sum_{j=1}^{n} x_j) = \alpha_i - \beta_i \sum_{j=1}^{n} x_j, \alpha_i \geq 0, \beta_i \geq 0 (i = 1, \cdots, n). \tag{2.5}$$

In this case, we take

$$\phi(x, y) := \left\langle \tilde{B}x - \alpha, y - x \right\rangle + y^T By - x^T Bx + h(y) - h(x) \tag{2.6}$$

where

$$\alpha^T = (\alpha_1, \alpha_2, \cdots, \alpha_n)$$

$$B := \begin{pmatrix} \beta_1 & 0 & 0 & \cdots & 0 \\ 0 & \beta_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \beta_n \end{pmatrix}, \tilde{B} := \begin{pmatrix} 0 & \beta_1 & \beta_1 & \cdots & \beta_1 \\ \beta_2 & 0 & \beta_2 & \cdots & \beta_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_n & \beta_n & \beta_n & \cdots & 0 \end{pmatrix}$$

and

$$h(x) := \sum_{i=1}^{n} h_i(x_i).$$

If for each firm, the cost function $h_i(.)$ is affine and defined by (2.4), then the equilibrium problem (EP), where $\Phi(x, y)$ is defined by (2.6) becomes the following mixed variational inequality problem (MVIP, for short):

$$\begin{cases} \text{find a point } x^* \in U \text{ such that} \\ \Phi(x^*, y) := \left\langle \tilde{B}x + \mu - \alpha, y - x \right\rangle + y^T By - x^T Bx \geq 0 \ \forall \ y \in U \end{cases} \tag{2.7}$$

Obviously, B is a symmetric positive semidefinite matrix. By [4], the mixed variational inequality problem (2.7) is equivalent to the following variational inequality:

$$\begin{cases} \text{find } x^* \in U \text{ such that} \\ \left\langle \tilde{Q}x^* + \mu - \alpha, y - x^* \right\rangle \geq 0 \ \forall \ y \in U \end{cases} \tag{VIP}$$

where

$$\tilde{Q} = \begin{pmatrix} 2\beta_1 & \beta_1 & \beta_1 & \cdots & \beta_1 \\ \beta_2 & 2\beta_2 & \beta_2 & \cdots & \beta_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_n & \beta_n & \beta_n & \cdots & 2\beta_n \end{pmatrix}$$

Thanks to its special structure, if $\beta_i > 0, \forall i = 1, \cdots, n,$ the affine variational inequality (VIP) is equivalent to the following strongly convex quadratic program (see [6]):

$$\min_{x \in U} \{\frac{1}{2} x^T \hat{Q} x + c^T x\} \tag{QP}$$

where

$$\hat{Q} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}$$

is symmetric positive definite and $c^T = (c_1, \cdots, c_n)$ with $c_i = (\mu_i - \alpha_i) / \beta_i (i = 1 \cdots, n).$

A mathematical programming approach for determining an equlibrium point to Cournot oligopolistic market model with convex cost function is proposed in [7].

3 A Vector Optimization Approach to Linear Cournot Model

In this section we describe the linear Cournot oligopolistic market model by using a vector optimization approach. In practice this approach can be used when we are interested in the total profit of all firms rather than the profit of each firm as in Nash equilibrium approach. It may happen that the total profit of the model at a Pareto efficient point is greater than that at the equilibrium point (see the illustrative example 3.1 below).

3.1 The Vector Optimization Cournot Model

As usual, for two vectors $x^T = (x_1, \cdots, x_n), \ y^T = (y_1, \cdots, y_n) \in \mathbb{R}^n$ we write $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, \cdots, n$ and $x \neq y.$

In the Cournot market to be considered in this section we suppose that the price and the cost functions of all firm are affine and given respectively by

$$p_i(\sigma):=p_i(\sum_{j=1}^{n}x_j)=\alpha_i-\beta_i\sum_{j=1}^{n}x_j, \alpha_i\geq 0, \beta_i\geq 0\ (i=1,\cdots,n)$$

and by

$$h_i(x_i)=\mu_i x_i+\xi_i, \mu_i\geq 0, \xi_i\geq 0\ (i=1,\cdots,n).$$

In this case, the profit function of firm $i(i=1,\cdots,n)$ can be rewritten as

$$f_i(x)=x_i p_i(x)-h_i(x_i)=-x^T C^i x+(\alpha_i-\mu_i)x_i-\xi_i, \tag{3.1}$$

where

$$C^i:=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_i & \beta_i & \beta_i & \cdots & \beta_i \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.2}$$

Let

$$f(x):=(f_1(x), f_2(x),\cdots, f_n(x))$$

then $f(.)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus the vector optimization problem to this Cournot model can be written as

$$\max_{x\in U}\{f(x)=(f_1(x), f_2(x),\cdots, f_n(x)\}. \tag{VP}$$

We recall [10] that a point $x^*\in U$ is a Pareto efficient solution to (VP) (or Pareto efficient point of *f* on *U*) if whenever $x\in U$ and $f_i(x)\geq f_i(x^*)$ for every $i=1,\cdots,n,$ then $f_i(x)=f_i(x^*)$ for every *i*. In what follows we are interested in finding an efficient Pareto solution to this vector optimization problem.

One of efficient and commonly used techniques to finding Pareto efficient solutions to the vector optimization problem (VP) is scalarization method. Actually, it defines the scalarization function

$$G(\lambda,x):=\sum_{i=1}^{n}\lambda_i f_i(x) \tag{3.2a}$$

where $\lambda_i>0(i=1,\cdots,n)$ are the weights. It is easy to see that if $\lambda_i>0(i=1,\cdots,n)$, then an optimal solution of the scalarized problem

$$\max\{G(\lambda,x):x\in U\} \tag{$VP1$}$$

is a Pareto efficient point of *f* over *U*. Particular, if $\lambda_i=1,\forall i=1,\cdots,n,$ then $G(\lambda,x):=\sum_{i=1}^{n}f_i(x).$ Thus, an optimal solution of the function $\sum_{i=1}^{n}f_i(x)$ over *U* is just the maximum of the total profit of the model.

To easy exposition, in the sequel, we take

$$F(\lambda, x) := -G(\lambda, x).$$

Then, instead of Problem (VP1), we can consider the problem

$$\min\{F(\lambda, x) : x \in U\}. \qquad (SVP)$$

since the solution-sets of these two problems coincide. From (3.1) and (3.2a) we have

$$F(\lambda, x) = x^T Q x + (\mu - \alpha)^T x + \xi, \qquad (3.3)$$

where

$$\alpha^T = (\alpha_1, \cdots, \alpha_n), \mu^T = (\mu_1, \cdots, \mu_n), \xi = \sum_{i=1}^n \xi_i;$$

$$Q := \begin{pmatrix} \beta_1\lambda_1 & \beta_1\lambda_1 & \cdots & \beta_1\lambda_1 \\ \beta_2\lambda_2 & \beta_2\lambda_2 & \cdots & \beta_2\lambda_2 \\ \cdots & \cdots & \cdots & \cdots \\ \beta_n\lambda_n & \beta_n\lambda_n & \cdots & \beta_n\lambda_n \end{pmatrix}.$$

Thus problem (SVP) is a quadratic program. Note that, since the matrix Q may not positive semidefinite, Problem (SVP) is not a convex program, and therefore finding a global optimal solution to the scalarized problem is a difficult task when the number of the variables is somewhat large (see [3,5]). However, thanks to the specific structure of the matrix Q, we can globally solve (SVP) by using a technique of the parametric linear programming. Namely, let $t := \sum_{i=1}^n x_i$, then

$$f_i(x) = (\alpha_i - \beta_i t)x_i - \mu_i x_i - \xi_i,$$

and the scalarization function now becomes

$$F(\lambda, t, x) := \sum_{i=1}^n \lambda_i(\beta_i t + \mu_i - \alpha_i)x_i + \xi.$$

Thus the scalarized problem (SVP) can be rewritten as the following:

$$\min\{F(\lambda, t, x) := \sum_{i=1}^n \lambda_i(\beta_i t + \mu_i - \alpha_i)x_i\} \qquad (LSVP)$$

$$\text{subject to} \begin{cases} x_1 + x_2 + \cdots + x_n = t; \\ 0 \le x_j \le b_j, j = 1, \cdots, p; \\ x_j \ge 0, j = p+1, \cdots, n; \\ t \ge 0. \end{cases}$$

In this problem, (t, x) are variables, and $\lambda_i > 0\ (i = 1, \cdots, n)$ are fixed. For each fixed $t \ge 0$, Problem (LSVP) is a linear program in variable x which can be rewritten as

$$\min\{F(\lambda,t,x):=\sum_{i=1}^{n}\lambda_i(\beta_i t+\mu_i-\alpha_i)x_i\} \qquad (LSVP(t))$$

$$\text{subject to}\begin{cases}x_1+x_2+\cdots+x_n=t;\\ 0\le x_j\le b_j,\ j=1,\cdots,p;\\ x_j\ge 0,\ j=p+1,\cdots,n.\end{cases}$$

Theorem 3.1. (i) For each fixed $t\ge 0$, Problem (*LSVP*(*t*)) has optimal solution.

(ii) If either $\beta_i>0$ or $\mu_i\ge\alpha_i$ for all $i=p+1,\cdots,n,$ then Problem (*LSVP*) has an optimal solution.

Proof.

(i) Obviously, because the feasible solution set of (*LSVP*(*t*)) is a bounded polyhedron.

(ii) Denote by *D* the set of feasible solutions to problem (*LSVP*). Obviously, *D* is a nonempty polyhedral convex set in $\mathbb{R}^{n+1}$. Suppose in contradiction that Problem (*LSVP*) has no optimal solution, then there exists a sequence $\{(t_k,x^k)\}\subset D$ such that

$$\lim_{k\to\infty}F(\lambda,t_k,x^k)=-\infty.$$

By definition of *F*, there exists an index *i* such that

$$p+1\le i_0\le n,\ \lim_{k\to\infty}\lambda_{i_0}(\beta_{i_0}t_k+\mu_{i_0}-\alpha_{i0})x_{i_0}^k=-\infty.$$

Since $\lambda_{i_0}>0$ and by the assumption, there exists $k_0\in N$ such that

$$(\beta_{i_0}t_k+\mu_{i_0}-\alpha_{i0})<0,\ \forall k>k_0$$

and

$$\lim_{k\to\infty}x_{i_0}^k=+\infty.$$

This is a contradiction, because when $\lim_{k\to\infty}x_{i_0}^k=+\infty$ then $\beta_{i_0}>0$ and $\lim_{k\to\infty}t_k=+\infty$, implies

$$\lim_{k\to\infty}(\beta_{i_0}t_k+\mu_{i_0}-\alpha_{i0})=+\infty.$$

Corollary 3.1. The vector optimization problem (*VP*) has always an efficient optimal solution.

3.2 Finding an Pareto Efficient Solution to the Model

In this section, we propose an algorithm for finding a Pareto efficient point of the Cournot model by using the parametric linear program (LSVP(t)).

For each fixed $t\ge 0$, let *D*(*t*) denote the feasible solution-set of Problem (LSVP(t)). Clearly, *D*(*t*) is a polyhedral convex set in $\mathbb{R}^n$ and $D(t)\ne\emptyset,\forall t\ge 0$. Moreover, the set of all vertices of *D*(*t*), denoted by *V*(*D*(*t*)), is always not empty for every $t\ge 0$. For each fixed $x(t)\in D(t),$ we define

$$J^{\text{active-R}}(x(t)) := \{j : x_j(t) = b_j, 1 \le j \le p\};$$
$$J^{\text{active-L}}(x(t)) := \{j : x_j(t) = 0, 1 \le j \le n\}.$$

From the theory of polyhedral convex sets in $\mathbb{R}^n$ (see e.g. [9]) the following proposition is immediate.

Proposition 3.1. For each fixed $t \ge 0$, $x(t)$, is a vertex of $D(t)$ if and only if the number of elements of the two sets $J^{\text{active-R}}(x(t))$ and $J^{\text{active-L}}(x(t))$ is not less than $n-1$.

For each $x(t) \in V(D(t))$, let $j_{x(t)}$ be an integer number such that $1 \le j_{x(t)} \le n$ and $\{1,\cdots,n\} \setminus \{j_{x(t)}\} \subseteq (J^{\text{active-R}}(x(t)) \bigcup J^{\text{active-L}}(x(t)))$ As usual, we call $j_{x(t)}$ a basis index of $x(t)$. If $j_{x(t)} \notin (J^{\text{active-R}}(x(t)) \bigcup J^{\text{active-L}}(x(t)))$ then $j_{x(t)} \bigcup (J^{\text{active-R}}(x(t)) \bigcup J^{\text{active-L}}(x(t))) = \{1,\cdots,n\}$. Specially, if $j_{x(t)} \in J^{\text{active-R}}$ or $j_{x(t)} \in J^{\text{active-L}}$ then $x(t)$ is nonregular, and it has more than one basic index. For each $x(t) \in D(t)$, we let:

$$J^{+}(x(t)) := J^{\text{active-R}}(x(t)) \setminus \{j_{x(t)}\};$$
$$J^{-}(x(t)) := J^{\text{active-L}}(x(t)) \setminus \{j_{x(t)}\}.$$

The following theorem is well known from the linear programming [2].

Theorem 3.3. For each fixed $t \ge 0$, there is at least one optimal solution $x^*(t)$ to problem (*LSVP* (*t*)), where $x^*(t) \in V(D(t))$.

For each $x(t) \in V(D(t))$, we define

$$\Delta_j(t) := c_{j_{x(t)}}(t) - c_j(t)(j = 1,\cdots,n, j \ne j_{x(t)}) \tag{3.4}$$

where $c_j(t) := \lambda_j(\beta_j t + \mu_j - \alpha_j), \forall j = 1,\cdots,n.$

Theorem 3.4. Let $t \ge 0$ be fixed. If $x^*(t) \in V(D(t))$ satisfies the following conditions

$$\begin{cases} \Delta_j(t) \ge 0, \forall j \in J^{+}(x(t)), \\ \Delta_j(t) \le 0, \forall j \in J^{-}(x(t)), \end{cases}$$

then $x^*(t)$ is an optimal solution to Problem (*LSVP*(*t*)).

Proof. Let $x(t)$ be any feasible solution to Problem (*LSVP*(*t*)). Since

$$\sum_{i=1}^{n} x_i(t) = \sum_{i=1}^{n} x_i^*(t) = t,$$

we have

$$x_{j_{x^*(t)}}(t) = x^*_{j_{x^*(t)}}(t) + \sum_{i \in J^{+}(x^*(t))} (b_i - x_i(t)) - \sum_{i \in J^{-}(x^*(t))} x_i(t).$$

By definition of $F(\lambda,t,x)$ and (3.4), a simple computation shows that

$$F(\lambda,t,x(x)) = F(\lambda,t,x^*(t)) + \sum_{i \in J^{+}(x^*(t))} \Delta_i(b_i - x_i(t)) - \sum_{i \in J^{-}(x^*(t))} \Delta_i x_i(t).$$

Obviously,

$$\begin{aligned} b_i - x_i(t) \geq 0, &\quad \forall i \in J^+(x^*(t)), \\ x_i(t) \geq 0, &\quad \forall i \in J^-(x^*(t)) \end{aligned}$$

Thus from the assumptions follows

$$F(\lambda, t, x(t)) \geq F(\lambda, t, x^*(t)), \forall x(t) \in D(t),$$

which means that $x^*(t)$ is an optimal solution to Problem (LSVP).

By the above definitions, for each fixed $t \geq 0$, $x(t) \in D(t)$ and index $1 \leq j \leq n$, if $x_i(t) = 0$ or $x_i(t) = b_i, \forall i = 1, \cdots, n, i \neq j$ then j is a basic index of $x(t)$.

For each fixed $1 \leq j \leq n$, we define the following sets:

$$V_{-j} := \{x^{-j} = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \in \mathbb{R}^{n-1}\},$$

where

$$x_i \in \{0, b_i\}(1 \leq i \neq j \leq p), x_i = 0(p+1 \leq j \neq j \leq n).$$

For each $x^{-j} \in V_{-j}$, we define a vector $x^j(t) \in \mathbb{R}^n$ respectively, where

$$x_i^j(t) = \begin{cases} x_i^{-j}, 1 \leq i \leq j-1; \\ t - \sum_{k \neq j}^{n} x_k^{-j}, i = j; \\ x_{i-1}^{-j}, j+1 \leq i \leq n. \end{cases} \tag{3.5}$$

Obviously, for each fixed $t \geq 0$, if $x^j(t) \in D(t)$ then $x^j(t) \in V(D(t))$. So, for each $x^{-j} \in V_{-j}$, we let

$$T(x^{-j}) := \{t \geq 0 : x(t) \in D(t)\}$$

and

$T^*(x^{-j}) := \{t \in T(x^{-j}) : x^j(t)$ is an optimal solution to Problem $(LSVP(t))\}$.

$$V_{-j}^* = \{x^{-j} \in V_{-j} : T^*(x^{-j}) \neq \emptyset\}.$$

Note that both $T^*(x^{-j})$ and V_{-j}^* can be empty sets.

Remark 3.1. From Theorem 3.1, it is easy to prove that there exists at least one index $j \in \{1, \cdots, n\}$ such that $V_{-j}^* \neq \emptyset$.

Suppose that $V_{-j}^* \neq \emptyset$ ($j \in \{1, \cdots, n\}$ is fixed), for each fixed $x^{-j} \in V_{-j}^*$, we define the subproblem of one-variable t

$$f^{x^{-j}} := \min\{F(\lambda, t, x^j(t)) : t \in T^*(x^{-j})\}. \tag{3.6}$$

Proposition 3.2. For each index $1 \leq j \leq n$, that $V_{-j}^* \neq \emptyset$. Suppose that at least one of the following two conditions is satisfied

(i) $\beta_j > 0$,

(ii) $\sum_{i \neq j}^{n} \lambda_i \beta_i + \lambda_j (\mu_j - \alpha_j) > 0$.

Then subprobem (3.6) has a unique optimal solution.

Proof. By (3.5),

$$x_i^j(t) = 0 \text{ or } x_i^j(t) = b_i, \forall i = 1, \cdots, n, i \neq j \tag{3.7}$$

and

$$x_j^j(t) = t - \sum_{i \neq j}^{n} x_i^j(t) := t - \delta, \tag{3.8}$$

where, by (3.7), δ does not depend on the variable *t*. By formulations in Problem (*LSVP*) and (3.8), a simple computation shows that:

$$\begin{aligned} F(\lambda, t, x(t)) &= \lambda_j (\beta_j t + \mu_j - \alpha_j) x_j(t) + \sum\nolimits_{i \neq j}^{n} \lambda_i (\beta_i t + \mu_i - \alpha_i) x_i \\ &= \beta_j t^2 + [\sum\nolimits_{i \neq j}^{n} \lambda_i \beta_i + \lambda_j (\mu_j - \alpha_j)] t + \gamma, \end{aligned}$$

where

$$\gamma = \sum_{i \neq j}^{n} [\lambda_i (\mu_i - \alpha_i) - \lambda_j (\mu_j - \alpha_j)] x_i.$$

Note that, by (3.7), γ is independent of *t*. So $F(\lambda, t, x(t))$ is expressed as a function of *t*. On the other hand, it is easy to show that $T^*(x^{-j})$ is a closed interval in $\mathbb{R}$. Thus, by definition of $T^*(x^{-j})$, itadmits minimizer at only one point $t \in T^*(x^{-j})$.

Obviously, suppose that, $t_{x^{-j}}$ is an optimal solution to the subproblem (3.6), then $x^j(t_{x^{-j}})$ is an optimal solution to Problem (LSVP$(t_{x^{-j}})$) with the minimal value $f^{x^{-j}}$.

For each fixed $1 \leq j \leq n$, let

$$f^j := \min\{f^{x^{-j}} : x^{-j} \in V_{-j}^*\}.$$

Suppose that it admits minimizer at $x^{*^{-j}}$, and t_j^* respectively is optimal solution to subproblem (3.6). By definition, $x^{*^j}(t_j^*)$ is an optimal solution respectively to Problem (*LSVP*(*t*)). As usual, we take $f^j = +\infty$ if $V_j^* = \emptyset$.

Corollary 3.2. Let

$$f^* := \min\{f^j : 1 \leq j \leq n\}.$$

Suppose that $f^* = f^{j_0}$ (minimum at $j = j_0$) and $x^{*^{j0}}(t_{j_0}^*)$ be a corresponding optimal solution to Probem (*LSVP*$(t_{j_0}^*)$). Then $(t_{j_0}^*, x^{*^{j0}}(t_{j_0}^*))$ is a global optimal solution to Problem (*LSVP*) and $x^{*^{j0}}(t_{j_0}^*)$ is a global optimal solution to Problem (SVP).

Proof. For each fixed $t \geq 0$, let $x(t)$ be a basic optimal solution to linear programming (*LSVP*(t)), and let f^t be the optimal value respectively. By definition, there exists an index $j_t (1 \leq j_t \leq n)$ such that j_t is the basic index of $x(t)$. Obviously, $x^{-j_t}(t) \in V^*_{-j_t}$ and

$$f^t \geq f^{j_t} \geq f^* = f^{j_0}. \tag{3.8}$$

Let f_* denote the minimal value of Problem (*LSVP*). By formulations of Problem (*LSVP*), Problem (*LSVP*(t)) and (3.8), we have

$$f_* = \min\{f^t : t \geq 0\} \geq f^{j_0}. \tag{3.9}$$

On the other hand, since $(t^*_{j_0}, x^{*j0}(t^*_{j_0}))$ is a feasible soulution of Problem (*LSVP*), we have

$$f^{j_0} \geq f_*.$$

Thus, $(t^*_{j_0}, x^{*j0}(t^*_{j_0}))$ is an optimal solution to Problem (*LSVP*), and therefore $x^{*j0}(t^*_{j_0}))$ is an optimal solution to Problem (SVP).

Now we are in a position to describe the algorithm.

Algorithm 3.1.

Let $j := 1;\ f^* := +\infty; x^* := 0.$

Iteration. j : Let $f^j := +\infty; x^{*j} := 0.$

Step 1. Choose a vector $x^{-j} = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \in V_{-j}$, where:

$$\begin{cases} x_i = 0 \text{ or } x_i = b_i (i = 1, \cdots, p, i \neq j), \\ x_i = 0\ (i = p+1, \cdots, n, i \neq j), \end{cases} \tag{3.9}$$

Step 2. Compute the set $T^*(x^{-j})$ by solving the inequalities:

$$\begin{cases} 0 \leq x_j(t) \leq b_j \text{ if } 1 \leq j \leq p, \\ x_j(t) \geq 0 \text{ if } p+1 \leq j \leq n. \\ \Delta_k(t) := c_j(t) - c_k(t) \leq 0\ \forall k = 1, \cdots, n, k \neq j, x_k = 0, \\ \Delta_k(t) := c_j(t) - c_k(t) \geq 0\ \forall k = 1, \cdots, p, k \neq j, x_k = b_k. \end{cases} \tag{3.10}$$

where $x_j(t) = t - \sum_{i \neq j} x_i$ and $c_i(t) = \lambda_i(\beta_i t + \mu_i - \alpha_i), \forall i = 1, \cdots, n.$

Step 3. If $T^*(x^{-j}) = \emptyset$ then let $V_{-j} := V_{-j} \setminus \{x^{-j}\}$ and go to Step 5. Otherwise, compute $f^{x^{-j}}, t_{x^{-j}}$, $x^j(t_{x^{-j}})$ by solving subproblem (3.6) with respectively $x^{-j} \in V_{-j}$.

Step 4. If $f^j > f^{x^{-j}}$ then:

$$f^j := f^{x^{-j}};$$

$$t^*_j := t_{x^{-j}};$$

$$x^{*j} := x^j(t_{x^{-j}}),$$

let $V_{-j} := V_{-j} \setminus \{x^{-j}\}$. Otherwise, let $V_{-j} := V_{-j} \setminus \{x^{-j}\}$.

Step 5. If $V_{-j} \neq \emptyset$ then go to *Step 1*. Otherwise: if $f^* > f^j$ then $f^* := f^j, x^* := x^{*^j}$.

Step 6. Let $j := j+1$. If $j \leq n$ then go to Iteration *j*.

Obviously, Algorithm 3.1 terminated, x^*, f^* is an optimal solution to Problem (*LSVP*).

From the description of the algorithm we see that, the efficiency of Algorithm 3.1 depends on the number x^{-j} in every $V_{-j}(j=1,\cdots,n)$. This number equals to 2^{p-1} if $j=1,\cdots,p$ and 2^p if $j=p+1,\cdots,n$. Thus, Algorithm 3.1 is efficient when *p* (the number of firms whose quantity of comodity is bounded) is not too large. This observation suggests the following relaxation algorithm that allows us to avoid computing all elements of V_{-j}.

Let *S* be a subset not empty of $\{1,\cdots,p\}$. We consider the following relaxed problem:

$$\min\{F(\lambda,t,x) := \sum_{i=1}^{n} \lambda_i(\beta_i t + \mu_i - \alpha_i)x_i\} \qquad (LSVP1)$$

$$\text{subject to}\begin{cases} x_1 + x_2 + \cdots + x_n = t; \\ 0 \leq x_j \leq b_j, j \in S; \\ x_j \geq 0, j \notin S; \\ t \geq 0. \end{cases}$$

Let x^* be an optimal solution to Problem (*LSVP*1). Obviously, if x^* is a feasible solution to Problem (*LSVP*) then x^* is an optimal solution to Problem (*LSVP*).

The relaxation algorithm now can be described as follows.

Algorithm 3.2.

Choose a subset *S* of $\{1,\cdots,p\}$, ordinarily, we take $S := \emptyset$.

Phase 1. Let $C := \emptyset$. Use Algorithm 3.1 to solve the relaxed Problem (*LSVP*1) to obtain its optimal solution x^*. For this relaxed problem form (3.9) in Step 1 of Algorithm 3.1 becomes

$$\begin{cases} x_i = 0 \text{ or } x_i = b_i (i \in S, i \neq j), \\ x_i = 0 (i \notin S, i \neq j). \end{cases} \qquad (3.9')$$

Phase 2. For all $1 \leq i \leq p, i \notin S$, if $x_i^* > b_i$, let $C := C \cup \{i\}$. If $C \neq \emptyset$ then $S := S \bigcup C$ and return to *Phase 1*. Otherwise, if $x_i^* \leq b_i$ for all $1 \leq i \leq p, i \notin S$ then terminate.

Obviously, Algorithm 3.2 terminates after finite iterations. In the worst case Algorithm 3.2. terminates with $S = \{1,2,\cdots,p\}$.

Illustrative example 3.1. We consider the Cournot model with three firms ($n=3$). The price, cost functions and strategy set for each firm are given as

$$\begin{cases} p_1(x) := 4.5 - 0.01(x_1 + x_2 + x_3), h_1(x_1) := 3x_1, U_1 := [0,30]; \\ p_2(x) := 6.4 - 0.015(x_1 + x_2 + x_3), h_2(x_2) := 5x_2, U_2 := [0,40]; \\ p_3(x) := 7.2 - 0.01(x_1 + x_2 + x_3), h_3(x_3) := 6x_3, U_3 := [0,+\infty). \end{cases}$$

In this model, the number of firms that have quantities are bounded is 2 ($p = 2$). Let $\lambda := (500, 200, 300)$ be the weight vector. In this case, Problem (LSVP(t)) then takes the form

$$\min\{F(\lambda,t,x) = (5t-750)x_1 + (3t-140)x_2 + (3t-360)x_3\}$$

$$\text{subject to}\begin{cases} x_1 + x_2 + x_3 = t \\ 0 \le x_1 \le 30 \\ 0 \le x_2 \le 40 \\ x_3 \ge 0 \\ t \ge 0 \end{cases}$$

Now, we describe detailed the solving this problem by Algorithm 3.2.

Let $S := \emptyset$.

Phase 1. (Solve the relaxed Problem ($LSVP1$) with $S := \emptyset$ by Algorithm 3.1). In this case, the relaxed problem is written as:

$$\min\{F(\lambda,t,x) = (5t-750)x_1 + (3t-140)x_2 + (3t-360)x_3\}$$

$$\text{subject to}\begin{cases} x_1 + x_2 + x_3 = t \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0, \\ t \ge 0. \end{cases}$$

Let $C := \emptyset;\ j := 1;\ f^* := +\infty;\ x^* := (0,0,0).$

Iteration $j = 1$. Let $f^1 := +\infty;\ x^{*^1} := (0,0,0).$

Step 1. Choose:

$$x^{-1} = (0,0) \in V_{-1} = \{(0,0)\}.$$

Step 2. Compute:

$$T^*(x^{-1}) = [0,195].$$

Step 3. Compute:

$$f^{x^{-1}} = -28125;\ t_{x^{-1}} = 75;\ x^1(t_{x^{-1}}) = (75,0,0).$$

Step 4. Compute:

$$f^1 := f^{x^{-1}} = -28125;\ t_1^* := t_{x^{-1}} = 75;\ x^{*^1} := x^1(t_{x^{-1}}) = (75,0,0);$$
$$V_{-1} := V_{-1} \setminus \{x^{-1}\} = \emptyset.$$

Step 5. Compute:

$$f^* := f^1 = -28125;\ x^* := x^{*^1} = (75,0,0).$$

Step 6. Let $j := 2$.

Iteration $j=2$: Let $f^2 := +\infty;\ x^{*^2} := (0,0,0)$.

Step 1. Choose:

$$x^{-2} = (0,0) \in V_{-2} = \{(0,0)\}.$$

Step 2. Compute:

$$T^*(x^{-2}) = \emptyset.$$

Step 3. Let

$$V_{-2} := V_{-2} \setminus \{x^{-2}\}.$$

Step 5. Check: $V_{-2} = \emptyset$.
Step 6. Let $j := 3$.

Iteration $j=3$. Let $f^3 := +\infty;\ x^{*^3} := (0,0,0)$.

Step 1. Choose:

$$x^{-3} = (0,0) \in V_{-3} = \{(0,0)\}.$$

Step 2. Compute:

$$T^*(x^{-3}) = [195, +\infty).$$

Step 3. Compute:

$$f^{x^{-3}} = 43875;\ t_{x^{-3}} = 195;\ x^1(t_{x^{-3}}) = (0,0,195).$$

Step 4. Compute:

$$f^3 := f^{x^{-3}} = 43875;\ t_3^* := t_{x^{-3}} = 195;\ x^{*^3} := x^1(t_{x^{-3}}) = (0,0,195);$$
$$V_{-3} := V_{-3} \setminus \{x^{-3}\} = \emptyset.$$

Step 5. Compute:

$$f^* := f^1 = -28125;\ x^* := x^{*^1} = (75,0,0).$$

Step 6. Let $j := 4$.

Phase 2. Update: $C := \{1\}; S := S \cup C = \{1\}$. Go to Phase 1 below.

Phase 1. (Solve the relaxed Problem (*LSVP*1) with $S := \{1\}$ by Algorithm 3.1). In this case, the relaxed problem can be rewritten as:

$$\min\{F(\lambda,t,x) = (5t-750)x_1 + (3t-140)x_2 + (3t-360)x_3\}$$

$$\text{subject to}\begin{cases} x_1 + x_2 + x_3 = t \\ 0 \le x_1 \le 30, \\ x_2 \ge 0,\ x_3 \ge 0, \\ t \ge 0 \end{cases}$$

Let $C := \emptyset; j := 1; f^* := +\infty; x^* := (0,0,0).$

Iteration $j = 1$. Let $f^1 := +\infty; x^{*^1} := (0,0,0).$

Step 1. Choose:

$$x^{-1} = (0,0) \in V_{-1} = \{(0,0)\}.$$

Step 2. Compute:

$$T^*(x^{-1}) = [0,30].$$

Step 3. Compute:

$$f^{x^{-1}} = -18000; t_{x^{-1}} = 30; x^1(t_{x^{-1}}) = (30,0,0).$$

Step 4. Compute:

$$f^1 := f^{x^{-1}} = -18000;\ t_1^* := t_{x^{-1}} = 30;\ x^{*^1} := x^1(t_{x^{-1}}) = (30,0,0);$$
$$V_{-1} := V_{-1} \setminus \{x^{-1}\} = \emptyset.$$

Step 5. Compute:

$$f^* := f^1 = -18000;\ x^* := x^{*^1} = (30,0,0).$$

Step 6. Let $j := 2.$

Iteration $j = 2$: Let $f^2 := +\infty;\ x^{*^2} := 0.$

Step 1. Choose:

$$x^{-21} = (0,0) \in V_{-2} = \{(0,0);(0,30)\}.$$

Step 2. Compute:

$$T^*(x^{-21}) = \emptyset.$$

Step 3. Let

$$V_{-2} := V_{-2} \setminus \{x^{-21}\} = \{(0,30)\}.$$

Step 5. Check: $V_{-2} = \{(0,30)\}$. Go to Step1.
Step 1. Choose:

$$x^{-22} = (0,30) \in V_{-2} = \{(0,30)\}.$$

Step 2. Compute:

$$T^*(x^{-22}) = \emptyset.$$

Step 3. Let

$$V_{-2} := V_{-2} \setminus \{x^{-22}\} = \emptyset.$$

Step 5. Check: $V_{-2} = \emptyset$.
Step 6. Let $j := 3$.

Iteration $j = 3$. Let $f^3 := +\infty;\ x^{*^3} := (0,0,0)$.

Step 1. Choose:

$$x^{-31} = (0,0) \in V_{-3} = \{(0,0),(30,0)\}.$$

Step 2. Compute:

$$T^*(x^{-31}) = [195,+\infty).$$

Step 3. Compute:

$$f^{x^{-31}} = 43875;\ t_{x^{-31}} = 195;\ x^{31}(t_{x^{-31}}) = (0,0,195).$$

Step 4. Compute:

$$f^3 := f^{x^{-31}} = 43875;\ t_3^* := t_{x^{-31}} = 195;\ x^{*^3} := x^{31}(t_{x^{-31}}) = (0,0,195);$$
$$V_{-3} := V_{-3} \setminus \{x^{-31}\} = \{(30,0)\}.$$

Step 5. Check: $V_{-3} = \{(30,0)\} \neq \emptyset$. Go to Step 1.
Step 1. Choose:

$$x^{-32} = (30,0) \in V_{-3} = \{(30,0)\}.$$

Step 2. Compute:

$$T^*(x^{-32}) = [30,195].$$

Step 3. Compute:

$$f^{x^{-32}} = -19200;\ t_{x^{-32}} = 50;\ x^{32}(t_{x^{-32}}) = (30,0,20).$$

Step 4. Compute:

$$f^3 := f^{x^{-32}} = -19200;\ t_3^* := t_{x^{-32}} = 50;\ x^{*^3} := x^{32}(t_{x^{-32}}) = (30,0,20);$$
$$V_{-3} := V_{-3} \setminus \{x^{-32}\} = \emptyset.$$

Step 5. Check: $V_{-3} = \emptyset$. Let $f^* := f^3 = -19200;\ x^* := x^{*^3} = (30,0,20)$.
Step 6. Let $j := 4$.

Phase 2. Since $C = \emptyset$, terminate.

Thus, $x^* = (30, 0, 20)$ is an optimal solution to the Problem (*SVP*), which is an efficient point (Pareto) of the model. It implies that the total profit of three firms is 44.

On the other hand, it is easy to check that $\bar{x} = (30, 12.2222, 38.8889)$ is the unique equilibrium point to the model. The total profit obtained by the equilibrium point $\bar{x}$ is 38.0309.

4 Computational Results and Experiences

We have tested the proposed algorithm for some groups of problems with random generated data. Each group contains 100 problems with the same size (*n*, *p*), but their data are different, since they are generated from different random processes. To compare with a local optimization algorithm, we use the MALAB to solve the nonconvex quadratic Problem (SVP). The results is reported in the Table 4.1. In the Table 4.1: *n* is the number of firms; *p* is the number of firms whose quantity of commodity is bounded; Average times 1 is the average times to solve one problem by Algorithm 3.2; Average times 2 is the average times to solve one problem by quadratic program in MALAB; Average error is the average error for one problem and Max - error is the maximal error for all 100 problems when comparing the quadratic code in MALAB with our code for solving same quadratic problem (SVP)

Table 4.1

Size: (n, p)	Average times 1.	Average times 2.	Average error	Max - error
10 - 10	0.0465	0.2879	-2.1608	-151.2683
20 - 20	0.1304	0.2643	-0.5213	-28.2098
30 - 30	0.2938	0.2403	-0.6027	-60.2274
50 - 50	0.8299	0.2771	-0.5801	-55.4158
60 - 60	1.3742	0.3954	-0.6766	-24.8064
70 - 65	1.4265	0.6725	-1.1875	-100.9106
70 - 70	1.7018	0.3442	-0.1428	-12.9841
80 - 70	1.4925	1.3701	-2.4237	-94.3451
100 - 50	0.4065	2.1861	-1.5073	-84.9459
100 - 70	0.9143	2.5646	-3.2416	-173.0152
100 - 90	2.8614	1.3273	-3.6793	-137.6352
100 - 100	3.9967	0.6162	-0.0650	-6.5026
200 - 100	1.2335	8.0889	-5.6636	-368.7423
200 - 150	4.4423	8.5445	-3.1932	-146.6946
200 - 200	20.8574	1.6518	-0.0057	-0.5724
300 - 50	0.9517	9.5784	-0.0391	-3.9104
300 - 100	1.0513	12.9821	-0.0000	-0.0000
300 - 200	7.2441	20.3511	-0.4132	-14.5284

Conclusion. The problem of finding an efficient point for Cournot oligopolistic market model is convered into a nonconvex quadratic problem. We have developed a relaxation algorithm for solving the latter problem. Computational results on a lot of randomly generated problems show efficiency of the proposed algorithm. We have also used a code in MATLAB to solve the tested problems that we have solved by the proposed algorithm. Since the optimal solutions that we have obtained by our algorithm are global, our computed optimal solutions are much better in the objective function value than those obtained by the MATLAB code.

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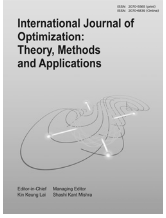

Stability Solution of Parametric Multiplicative Extremal Problems

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Abstract. The problem**,** In this paper, is solving mathematical programming problem in which the objective function is multiplicated of some real valued functions contain certain parameters. Also the stability of solution is discussed. The approach, dealing with this problem, is constructing an algorithm to obtain the solution and determining the set of all parameters corresponding this solution. The main result, in this work is there is same solution for many extremal problems change parametrically.

Keywords: multiplicative programming, parametric study, stability.

1 Introduction

It is well known the multiplicative extremal problems generally possess many local optimal solutions that are not globally optimal. This pushes many researchers to discuss the globality for this kind of problems which have important applications in many areas such as bond portfolio optimization [2], economic analyses [3] and very large system integrated (VLSI) ship design [5].

In case of multiplying two real functions, Tuy and Tam [6] presented a polyhedral annexation algorithm for finding the global optimal solution. Also, Kuno [4] proposed a branch-bounded algorithm, Aneja, et al. [1] proposed an outer approximation algorithm. Youness and Rokaya introduced

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in [7] an equivalence parametric formulation for multiplicative extremal problems and constructed an algorithm to find the global solution.

In this paper, we interest with global solution of parametric multiplicative extremal problem and the set of all parameters that make this solution stable.

2 Description and Derivation of the Approach

Consider a parametric multiplicative extremal problem

$$(PMP)\begin{cases} \min \quad \prod_{i=1}^{p} f_i(x,\lambda) \\ S.t. \\ M=\{x\in R^n : g_r(x)\le o,\, r=1,2,\cdots,m\} \\ \end{cases}$$

Where $f_i(x,\lambda)$, $i=1,2,\cdots,p$ are convex on $M\times R^l$, $\lambda\in R^l$ is the vector parameter in parametric space and $g_r(x), r=1,2,\cdots,m$ are convex functions on M

In case $P = 2$, we have the problem

$$(PMP)_2\begin{cases} f_1(x,\lambda)f_2(x,\lambda) \\ s.t. \\ M=\{x\in R^n : g_r(x)\le 0, r=1,2,\cdots,m\}, \end{cases}$$

Let $f_1 > 0$ on M for each λ, and convert the problem into the following two problems:

$$P_1\begin{cases} \eta(\delta,\lambda)=\min\limits_{x\in M} f_1(x,\lambda) \\ s.t. \\ f_2(x,\lambda)\le\delta, \\ \delta\in\Delta=\left\{\delta\in R: M\bigcap\left\{x\in R^n : f_2(x,\lambda)\le\delta\right\}\ne\varphi\right\} \end{cases}$$

and

$$P_2\begin{cases} \min\eta(\delta,\lambda)\delta \\ s.t. \\ \delta\in\Delta, \end{cases}$$

Lemma 1. If $f_2(x,\lambda)$ is continuous and convex on $M\times R^l$. Then the set Δ is compact.

Proof: It is clear, from continuity of f_2 on $M\times R^l$, that the set Δ is closed. Now we will prove it is bounded. Suppose the set Δ is unbounded, then there is a direction $s, |s|=1$ such that $\delta+\alpha s$ $\in\Delta$ for each $\alpha\ge 0$. Since $|s|=1$, either $s = 1$ or $s = -1$. Now consider the case $s = 1$:

Since $f_2(x,\lambda)$ is convex on M for each λ so the level sets

$$\Gamma(\delta) = \{x \in M : f_2(x,\lambda) \le \delta\}$$

is a subset of the level set

$$\Gamma(\delta+\alpha) = \{x \in M : f_2(x,\lambda) \le \delta+\alpha, \alpha > 0\}.$$

Now assume $\overline{x} \in \Gamma(\delta), \hat{x} \in \Gamma(\delta+\alpha)$ and $\hat{x} \notin \Gamma(\delta)$. Thus, for $\nu \in (0,1)$, we have

$$\nu f_2(\overline{x},\lambda) \le \nu\delta,$$

$$(1-\nu) f_2(\hat{x},\lambda) \le (1-\nu)(\delta+\alpha)$$

and

$$f_2(\nu\overline{x} + (1-\nu)\hat{x}, \lambda) \le \delta + (1-\nu)\alpha$$

By supposing, without lost of generality, $\alpha = \nu$, then

$$f_2(\nu\overline{x} + (1-\nu)\hat{x}, \lambda) \le \delta + \nu(1-\nu)$$

By letting v tends to zero, we get

$$f_2(\hat{x},\lambda) \le \delta$$

which is a contradiction.

Also, in the case $s = -1$, we can obtain a contradiction in the same manner. Hence the set Δ is bounded.

Theorem 2. If $f_2(x,\lambda)$ is continuous on $M \times R^l$ and δ^0 is an optimal solution of problem P_2 corresponding to λ^0. Then the optimal value of the objective function of problem $(PMP)_2$ is given by $\eta(\delta^0,\lambda^0)\delta^0$.

Proof: Let x^0 be an optimal solution of problem $(PMP)_2$, then

$$\delta^0 = f_2(x^0,\lambda^0) \in \Delta$$

and

$$\begin{gathered} f_1(x^0,\lambda^0)\, f_2(x^0,\lambda^0) = f_1(x^0,\lambda^0)\, \delta^0 \ge \min\{ f_1(x,\lambda^0)\, \delta^0 : \\ x \in M,\, f_2(x,\lambda^0) \le \delta^0 \} \ge \eta(\delta^0,\lambda^0)\, \delta^0 \end{gathered} \tag{1}$$

Due to problem P_2 and the definition of η, for the given δ^0, there exists an

$$x' \in M \cap \{x \in R^n : f_2(x,\lambda^0) \le \delta^0\}$$

with

$$\begin{aligned} \eta\,(\delta^0,\lambda^0)\,\delta^0 &= f_1(x',\lambda^0)\,\delta^0 \ge f_1(x',\lambda^0)\, f_2(x',\lambda^0) \\ &\ge f_1(x^0,\lambda^0)\, f_1(x^0,\lambda^0) \end{aligned} \tag{2}$$

From (1) and (2), we get

$$f_1(x^0, \lambda^0)\ f_2(x^0, \lambda^0) = \eta\ (\delta^0, \lambda^0)\ \delta^0$$

Denote Δ^0 the set of optimal solutions for the problem $P_2, X(\delta)$ the set of optimal solutions of problem P_1 corresponding to $\delta \in \Delta$ and X^* the set of optimal solutions for the problem $(PMP)_2$ corresponding to λ^0.

Theorem 3. $X^* = \bigcup_{\delta \in \Delta^0} X(\delta)$.

Proof: $x^0 \in \bigcup_{\delta \in \Delta^0} X(\delta)$ implies $x^0 \in X(\delta)$ for at least $\delta^0 \in \Delta, x^o$ fulfills the constraints in problem P_1 for $\delta = \delta^0$ and $\eta(\delta^0, \lambda^0) = f_1(x^0, \lambda^0)$. So x^0 satisfies in particular the constraints in problem $(PMP)_2$ and

$$f_1(x^0, \lambda^0)\ f_2(x^0, \lambda^0) = \eta\ (\delta^0, \lambda^0)\ \delta^0.$$

Hence by theorem 2, $x^0 \in X^*$.

Let $x^0 \in X^*$ and $f_2(x^0, \lambda^0) = \delta'$. By theorem 2 we get $x^0 \in X(\delta')$ with $\eta(\delta', \lambda^0) = f_1(x^0, \lambda^0)$, i.e., $\delta' \in \Delta^0$. Hence $x^0 \in \bigcup_{\delta \in \Delta^0} X(\delta)$ and $X^* = \bigcup_{\delta \in \Delta^0} X(\delta)$.

3 Stability Set of the Solution

Let $\lambda^0 \in R^l$ and $x^0 \in M$ be an optimal solution of the problem $(PMP)_2$ corresponding to λ^0. Stability set of the solution x^0 is denoted $S(x^0)$ and is defined as:

$$S(x^0) = \left\{ \lambda \in R^l : x^0 \text{ is an optimal solution of problem } (PMP)_2 \right\}.$$

To determine the set $S(x^0)$ we must solve the problem firstly to obtain the solution x^0 and secondly go to the determination of all parameters corresponding this solution. To do this follow the following algorithm.

4 The Algorithm

The algorithm for solving the problem $(PMP)_2$ and determining the stability set of parameters correspond to the solution can be summarized in the following steps.

1) Choose $x^k \in M$ and $\lambda^k \in R^l$ and compute $\delta^k = f_2(x^k, \lambda^k)$.
2) Determine the points $x \in M$ such that $f_2(x, \lambda^k) \leq \delta^k$. Denote this set N.
3) Determine the solution of P_1 corresponding to δ^k, i.e, the solution of the problem

$$\min_{x \in N} f_1(x, \lambda^k).$$

Denote this solution x^{k+1}.

4) Compute $\delta^{k+1} = f_2(x^{k+1}, \lambda^k)$.
5) If δ^{k+1} is the solution of P_2 then x^{k+1} is the optimal solution of $(PMP)_2$ corresponding to λ^k and hence go to step 6. Otherwise go to step 2.

6) Solve the Kuhn-Tucker equations in λ to obtain $S(x^k)$,

$$\begin{gathered}\nabla f_1(x^k,\lambda)+\mu_0\nabla f_2(x^k,\lambda)+\mu_r\sum_{r=1}^{m}\nabla g_r(x^k)=0\\ \mu_0\left[f_2(x^k,\lambda)-\delta^k\right]=0,\\ \mu_r g_r(x^k)=0,\ r=1,2,\cdots,m\\ \mu_0,\mu_r\geq 0,\ r=1,2,\cdots,m\end{gathered}$$

5 Illustrative Example

Determine the solution and the stability set for this solution of the following multiplicative program

$$(MP)\begin{cases}\min f_1(x,\lambda)f_2(x,\lambda)\\ s.t.\\ M=\{x\in R^2:0\leq x\leq 1\},\end{cases}$$

where $x,\lambda\in R^2$ and $f_1=(\lambda_1x_1-2)^2, f_2=x_2-2\lambda_2$.
choose $x=(\frac{1}{2},\frac{1}{2})$ and $(\lambda_1^0,\lambda_2^0)=(1,1)$, then $f_2(x^0,\lambda^0)=\delta^0=-\frac{3}{2}$. The set N is

$$\begin{aligned}N&=\left\{x\in M:(x_2-2)\leq-\frac{3}{2}\right\}\\ &=\left\{(x_1,x_2):0\leq x_1\leq 1\ ,\ 0\leq x_2\leq\frac{1}{2}\right\}.\end{aligned}$$

Solve the problem

$$\begin{aligned}&\min(x_1-2)^2\\ &s.t.\\ &x\in N\end{aligned}$$

Which it has the set of solutions $\left\{x\in N:x=(1,t),t\in[0,\frac{1}{2}]\right\}$.
Compute $\delta^1=f_2(x^1,1)=t-2,\ x^1\in N,$ and determine the set

$$\Delta=\left\{\delta\in R:M\cap\left\{x\in R^2:f_2(x,\lambda^0)\leq t-2\right\}\neq\varphi\right\}$$

We can verify that $\delta'=-2,\ x=(0,0)$ is the solution of the problem

$$\begin{aligned}&\min\delta(x_1-2)^2\\ &s.t.\\ &0\leq x\leq 1,\\ &\delta\in\Delta=\left\{\delta\in R:M\cap\left\{x\in R^2:x_2-2\leq-2\right\}\neq\varphi\right\}\end{aligned}$$

From Kuhn-Tucker necessary conditions we can obtain the system of equations

$$
\begin{gathered}
-4\lambda_1 + \mu_1 - \mu_2 = 0 \\
-2\lambda_2\mu_0 + \mu_3 - \mu_4 = 0 \\
\mu_0(2 - 2\lambda_2) = 0 \\
\mu_1 \geq 0,\ \mu_2 \geq 0,\ \mu_3 \geq 0,\ \mu_4 \geq 0,\ \mu_0 \geq 0
\end{gathered}
$$

By solving this system in λ_1, λ_2 we obtain the set $S(0,0)$ as follows

$$S(0,0) = \left\{ (\lambda_1, \lambda_2) : \lambda_1 = \frac{1}{4}\mu_1,\ \lambda_2 = \frac{1}{2}\frac{\mu_4}{\mu_0},\ \mu_0 \neq 0,\ \mu_1, \mu_4 \geq 0 \right\}$$

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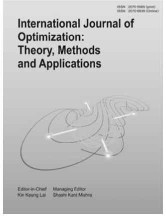

An Algorithm for Solving Fixed Charge Bi-criterion Indefinite Quadratic Transportation Problem with Restricted Flow

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Abstract. In the present paper a fixed charge bi-criterion quadratic transportation problem with restriction on the total flow is studied. An algorithm to find the efficient cost-time trade off pairs in a fixed charge bi-criterion quadratic transportation problem is presented. A related fixed charge bi-criterion quadratic transportation problem is formulated and the efficient cost-time trades off pairs to the given problem are shown to be derivable from this related problem. The algorithm is illustrated with the help of a numerical example.

Keywords: transportation problem, cost-time trade off, bi-criterion quadratic transportation problem, restricted flow.

Mathematics Subject Classification 1991: Primary: 90B06; Secondary: 90C29

1 Introduction

The fixed charge problem was originally formulated by G. B. Dantzig and W. Hirsch [8] in 1954. Several procedures have been developed for solving fixed-charge transportation problems [2-3, 5-9, 12-17]. Also the time minimizing transportation problems have been studied many times [10, 11]. Vari-

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ous types of algorithms for finding time-cost trade off pairs in a classical transportation problem have been developed in [5-7]. Basu *et al*. [6] in 1994 developed an algorithm for the optimum time-cost trade off pair in a fixed charge bi-criterion transportation problem. Thirwani *et al.* [17] in 1997 developed an algorithm for finding time- cost trade off pairs in a fixed charge bi-criterion transportation problem with restricted flow. Later in 2001 Arora and Khurana [4] studied the indefinite quadratic transportation problem. Afterwards, the three dimensional fixed charge indefinite quadratic transportation problem was studied by Arora and Khurana [5] in 2004. They also developed cost-time trade off pairs for the problem.

2 Problem Formulation and Theoretical Development

We know linear functions are of the type most useful and widely used in modelling of mathematical optimization problems. Also quadratic functions and quadratic problems are the least difficult to handle out of all non-linear programming problems. A fair number of functional relationships occurring in the real world are truly quadratic. For example kinetic energy carried by a rocket or an atomic particle is proportional to the square of its velocity. In statistics, the variance of a given sample of observations is a quadratic function of the values that constitute the sample. So there are countless other non-linear relationships occurring in nature, capable of being approximated by quadratic functions.

Consider the Fixed Charge Bi-criterion Quadratic Transportation Problem given by

$$\text{Minimize} \quad \left\{ \left(\sum_{i\in I} \sum_{j\in J} c_{ij} x_{ij} \right) \left(\sum_{i\in I} \sum_{j\in J} d_{ij} x_{ij} \right) + \sum_{i\in I} F_i \,,\ \underset{\substack{i\in I \\ j\in J}}{Max} \left[t_{ij} \,|\, x_{ij} > 0 \right] \right\}$$

$$\text{Subject to} \quad \sum_{j\in J} x_{ij} = a_i \qquad i\in I$$

$$\sum_{i\in I} x_{ij} = b_j \qquad j\in J$$

$$x_{ij} \geq 0 \qquad i\in I,\ j\in J$$

where $I = \{1, 2, \cdots, m\}$ is the set of origins

$J = \{1, 2, \cdots, n\}$ is the set of destinations

x_{ij} = the quantity transported from the ith origin to the jth destination

c_{ij} = per unit cost in transporting goods from the ith origin to jth destination

d_{ij} = per unit depreciation cost in transporting goods from the ith origin to jth destination

t_{ij} = the time of transporting the product from the ith origin to the jth destination which is independent of the amount of commodity transported , so long as $x_{ij} > 0$

F_i = the fixed cost (or fixed charge) associated with the ith origin

a_i = the amount available at the ith origin

b_j = the demand at the jth destination.

The total flow in the problem is $\sum_{i\in I} a_i = \sum_{j\in J} b_j$

In the above problem the total transportation cost of transporting one unit from ith origin to jth destination is $\sum_i \sum_j c_{ij} x_{ij}$, but while transporting goods from one origin to the other destination, some fraction of goods get damaged so the total cost of damaged goods is $\sum_i \sum_j d_{ij} x_{ij}$. Our aim is to minimize the two costs simultaneously; therefore we consider the product of two costs i.e., $\left(\sum_i \sum_j c_{ij} x_{ij}\right)$ $\left(\sum_i \sum_j d_{ij} x_{ij}\right)$. Also we need to minimize the fixed cost associated with ith origin and the time of transportation from ith origin to jth destination.

If in the above problem the total availability is not equal to the total demand, then some of the source and/or destination constraints are satisfied as inequalities. Sometimes, situations arise when one wishes to keep reserve stocks at the sources for emergencies, thereby restricting the total transportation flow to a known specified level, say P (<min ($\sum_{i\in I} a_i, \sum_{j\in J} b_j$)). This flow constraint changes the structure of the transportation problem. The resulting fixed charge bi-criterion quadratic transportation problem with restricted flow is

$$(P_1)\text{: Minimize } \left\{\left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}\right)+\sum_{i\in I}F_i\,,\ \underset{\substack{i\in I\\ j\in J}}{Max}\left[t_{ij}\,|\,x_{ij}>0\right]\right\}$$

$$\text{Subject to } \left.\begin{aligned} &\sum_{j\in J} x_{ij} \le a_i, \quad i\in I\\ &\sum_{i\in I} x_{ij} \le b_j, \quad j\in J\\ &\sum_{i\in I}\sum_{j\in J} x_{ij} = P(<\text{Min }(\sum_{i\in I} a_i, \sum_{j\in J} b_j))\\ &x_{ij}\ge 0, \quad i\in I,\, j\in J \end{aligned}\right\} \quad \cdots \qquad (1)$$

In order to solve the above problem (P_1) we separate it into two problems (P_1') and (P_1'') where

(P_1') Minimize the cost function $\left\{\left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}\right)+\sum_{i\in I}F_i\right\}$ subject to (1)

and

(P_1'') Minimize the time function $\left\{\underset{\substack{i\in I\\ j\in J}}{\text{Max}}\left[t_{ij}\,|\,x_{ij}>0\right]\right\}$ subject to (1)

To formulate $F_i\{i=1,2,\cdots,m\}$ we assume that $F_i\{1,2,\cdots,m\}$ has p number of steps so that

$$F_i=\sum_{l=1}^{p}\delta_{il}F_{il}, \quad i=1,2\cdots,m$$

where $\delta_{il}=1$ if $\sum_{j=1}^{n} x_{ij} > K_{il}, i=1,2,\cdots,m; l=1,2,\cdots,p$

$= 0$ otherwise

Here $0=K_{i1}<K_{i2}<\cdots<K_{ip}$

Also $K_{i1},K_{i2},\cdots,K_{ip}(i=1,2,\cdots,m)$ are constants and $F_{il}(l=1,2,\cdots,p;i=1,2,\cdots,m)$ are fixed costs.

This flow constraints in the problem (P_1') implies that a total $\left(\sum_{i\in I} a_i - P\right)$ of the source reserves has to be kept at the various sources and a total $\left(\sum_{j\in J} b_j - P\right)$ of destination slacks is to be retained at the various destinations. Therefore, an extra destination to receive the source reserves and an extra source to fill up the destinations slacks are introduced. Hence the related fixed charge bi-criterion quadratic transportation problem (P_2) associated with fixed charge bi-criterion quadratic transportation problem (P_1) is

$$(P_2):\quad \text{Minimize}\ \left\{\left(\sum_{i\in I}\sum_{j\in J} c'_{ij}x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d'_{ij}x_{ij}\right)+\sum_{i\in I}F_i\ ,\ \underset{\substack{i\in I\\ j\in J}}{Max}\left[t'_{ij}\,|\,x_{ij}>0\right]\right\}$$

$$\text{Subject to}\quad \left.\begin{array}{ll} \sum_{j\in J'} x_{ij} = a'_i, & i\in I' \\ \sum_{i\in I'} x_{ij} = b'_i, & j\in J' \\ \sum_{i\in I'}\sum_{j\in J'} x_{ij} = P & \\ x_{ij}\ge 0, & i\in I',\ j\in J' \end{array}\right\}\ \cdots \qquad (2)$$

where

$$\begin{aligned}
&I' = \{1,2,\cdots,m+1\} = I\cup\{m+1\}\\
&J' = \{1,2,\cdots,n+1\} = J\cup\{n+1\}\\
&a'_i = a_i \quad i\in I,\quad a'_{m+1} = \left(\sum_{j\in J} b_j - P\right)\\
&b'_j = b_j \quad j\in J,\quad b'_{n+1} = \left(\sum_{i\in I} a_i - P\right)\\
&c'_{ij} = c_{ij},\quad (i,j)\in I\times J\\
&d'_{ij} = d_{ij},\quad (i,j)\in I\times J\\
&t'_{ij} = t_{ij},\quad (i,j)\in I\times J\\
&c'_{i,n+1} = c'_{m+1,j} = 0,\quad i\in I, J\in J\\
&d'_{i,n+1} = d'_{m+1,j} = 0,\quad i\in I, J\in J\\
&t'_{i,n+1} = t'_{m+1,j} = 0,\quad i\in I, J\in J\\
&c'_{m+1,n+1} = M = d'_{m+1,n+1}\\
&t'_{m+1,n+1} > \underset{\substack{i\in i\\ j\in j}}{\mathrm{Max}}\left\lfloor t_{ij}\,|\,x_{ij}>0\right\rfloor
\end{aligned}$$

where M is a large positive number.

(P_2) is separated into two problems (P'_2) and (P''_2) where

$$(P'_2):\quad \text{Minimize}\ Z = \left\{\left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}\right)+\sum_{i\in I}F_i\right\}$$

Subject to (2)

$$(P_2''):\quad \text{Minimize } T = \underset{\substack{i\in I\\ j\in J}}{\text{Max}}\left\lfloor t_{ij} \mid x_{ij} > 0 \right\rfloor$$

Subject to (2)

Consider the transportation problem where the objective function is the product of two linear functions.

$$\text{(QTP):}\quad \text{Minimize } Z = \left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}\right)\left(\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}\right)$$

$$\text{Subject to } \sum_{j\in J} x_{ij} \le a_i \qquad \forall\ i\in I$$

$$\sum_{i\in I} x_{ij} \le b_j \qquad \forall\ j\in J$$

$$x_{ij} \ge 0, \qquad \forall\ i\in I, j\in J$$

Note: The objective function being a quasiconcave function will have its basic feasible solution at its extreme points.

Theorem 1. Let $X = \{x_{ij}\}$ be a basic feasible solution of (QTP) with basis matrix B. Then it will be an optimal basic feasible solution if

$$\begin{aligned} R_{ij} &\ge 0 && \forall \text{ cells } (i,j)\notin B \\ &= 0 && \forall \text{ cells } (i,j)\in B \end{aligned}$$

where $R_{ij} = \theta_{ij}(z'_{ij} - d_{ij})(z_{ij} - c_{ij}) - Z_1(z'_{ij} - d_{ij}) - Z_2(z_{ij} - c_{ij})$

$$\begin{aligned} u_i + v_j &= z_{ij} && \forall \text{ cells } (i,j)\notin B \\ u'_i + v'_j &= z'_{ij} && \forall \text{ cells } (i,j)\notin B \end{aligned}$$

$$\text{Also}\qquad \left.\begin{aligned} u_i + v_j &= c_{ij} && \forall \text{ cells } (i,j)\in B \\ u'_i + v'_j &= d_{ij} && \forall \text{ cells } (i,j)\in B \end{aligned}\right\} \cdots \qquad (3)$$

Z_1 = the value of $\sum_i\sum_j c_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis *B*.

Z_2 = the value of $\sum_i\sum_j d_{ij}x_{ij}$ at the current basic feasible solution corresponding to the basis *B*.

and θ_{ij} is the level at which a non basic cell (i, j) enters the basis replacing some basic cell of *B*.

Note. u_i, v_j, u'_i, v'_j corresponds to dual variables and are determined by using equations (3) and taking one of the u_i's or v_j's and u'_i's or v'_j's as zero.

Proof: Let Z^0 be the objective function value of the problem (QTP).

Let $Z^0 = Z_1Z_2$

Let $\hat{Z}$ be the objective function value at the current basic feasible solution $\hat{X} = \{x_{ij}\}$ corresponding to the basis B obtained on entering the cell (i, j) into the basis.

Then $\qquad \hat{Z} = \lfloor Z_1 + \theta_{ij}(c_{ij} - z_{ij}) \rfloor \lfloor Z_2 + \theta_{ij}(d_{ij} - z'_{ij}) \rfloor$

Now,

$$\begin{aligned}\hat{Z} - Z^0 &= Z_1Z_2 + Z_1\theta_{ij}(d_{ij} - z'_{ij}) + Z_2\theta_{ij}(c_{ij} - z_{ij}) + \theta_{ij}^2(c_{ij} - z_{ij})(d_{ij} - z'_{ij}) - Z_1Z_2 \\ &= Z_1\theta_{ij}(d_{ij} - z'_{ij}) + Z_2\theta_{ij}(c_{ij} - z_{ij}) + \theta_{ij}^2(c_{ij} - z_{ij})(d_{ij} - z'_{ij}) \\ &= \theta_{ij}\lfloor Z_1(d_{ij} - z'_{ij}) + Z_2(c_{ij} - z_{ij}) + \theta_{ij}(c_{ij} - z_{ij})(d_{ij} - z'_{ij}) \rfloor\end{aligned}$$

This basic feasible solution will give an improved value of Z if $\hat{Z} < Z^0$

i.e., if $\theta_{ij}\lfloor Z_1(d_{ij} - z'_{ij}) + Z_2(c_{ij} - z_{ij}) + \theta_{ij}(c_{ij} - z_{ij})(d_{ij} - z'_{ij}) \rfloor < 0$

Since $\theta_{ij} \geq 0$

$$\therefore \quad Z_1(d_{ij} - z'_{ij}) + Z_2(c_{ij} - z_{ij}) + \theta_{ij}(c_{ij} - z_{ij})(d_{ij} - z'_{ij}) < 0 \tag{4}$$

Therefore one can move from one basic feasible solution to another basic feasible solution on entering the cell (i, j) into the basis for which condition (4) is satisfied.

It will be an optimal basic feasible solution if

$$R_{ij} = \theta_{ij}(z'_{ij} - d_{ij})(z_{ij} - c_{ij}) - Z_1(z'_{ij} - d_{ij}) - Z_2(z_{ij} - c_{ij}) \geq 0$$

Also it can be easily seen that $R_{ij} = 0 \ \forall$ cells $(i, j) \in B$

Definition—Corner Feasible Solution: A basic feasible solution $\{y_{ij}\}$, $i \in I'$, $j \in J'$ to (P_2) is called a corner feasible solution (cfs) if $y_{m+1,n+1} = 0$

Theorem 2. Every corner feasible solution of (P_2) provides a basic feasible solution to (P_1) and conversely.

Proof: let $\{y_{ij}\}$ be a cfs to (P_2). Define $x_{ij} = y_{ij}, (i, j) \in I \times J$

$\{x_{ij}\}$ so defined can be established to be a basic feasible to (P_1).

Conversely, Given $\{x_{ij}\}$ to be a basic feasible solution to (P_1) then

$\{y_{ij}\}, (i, j) \in I' \times J'$

where

$$I' = \{1, 2, \cdots, m+1\}$$
$$J' = \{1, 2, \cdots n+1\}$$

defined by the transformation

$$\begin{aligned} y_{ij} &= x_{ij}, && (i, j) \in I \times J \\ y_{i,n+1} &= a_i - \sum_{j \in J} x_{ij}, && i \in I \\ y_{m+1,j} &= b_j - \sum_{i \in I} x_{ij}, && j \in J \end{aligned}$$

$$y_{m+1,n+1} = 0$$

can be shown to be a cfs to (P_2).

Remark 1. A cfs of (P_2) is also a cfs of (P_2').

Remark 2. The value of the objective function of (P_2') at a corner feasible solution is equal to the value of the objective function of (P_1') at its corresponding basic feasible solution.

Value of the objective function of (P_2') is

$$\begin{aligned}
&= \sum_{i\in I'}\sum_{j\in J'} c'_{ij} y_{ij} + \sum_{i\in I} F_i \\
&= \sum_{i\in I}\sum_{j\in J} c'_{ij} y_{ij} + \sum_{j\in J'} c'_{m+1,j} y_{m+1,j} + \sum_{i\in I'} c'_{i,n+1} y_{i,n+1} + \sum_{i\in I} F_i \\
&= \sum_{i\in I}\sum_{j\in J} c_{ij} y_{ij} + \sum_{i\in I} F_i \qquad (\because c'_{m+1,j} = c'_{i,n+1} = 0,\ y_{m+1,n+1} = 0) \\
&= \text{Value of the objective function of } (\mathrm{P}_1')
\end{aligned}$$

Remark 3. A non-corner feasible solution to (P_2) cannot provide a feasible solution to (P_1).

Theorem 3. An optimal solution to (P_2') has to be a corner feasible solution.

Proof: If possible, let there exists an optimal solution $\{y'_{ij}\}$ to (P_2') which is not a cfs. Let the optimal cost for (P_2') be C^1, which is a large positive number. Now, consider an optimal solution $\{x^0_{ij}\}$ say to (P_1'), with corresponding optimal cost C^0. Let $\{y^0_{ij}\}$ be the corresponding cfs to (P_2'). The cost corresponding to the cfs $\{y^0_{ij}\}$ is also C^0. Clearly $C^0 < C^1$ which contradicts the fact that $\{y'_{ij}\}$ is an optimal solution to (P_2'). So, no non-cfs to (P_2') can be an optimal solution.

Theorem 4. There is a one to one correspondence between optimal solution to (P_1') and optima among the corner feasible solutions to (P_2').

Proof: Let $\{x^0_{ij}\}$ be an optimal solution to (P_1') yielding optimal cost C^0 and $\{y^0_{ij}\}$ be the corresponding cfs to (P_2'). The cost corresponding to the solution $\{y^0_{ij}\}$ is C^0. If possible let $\{y^0_{ij}\}$ be not an optimal cfs to (P_2'). So there exists an optimal cfs y^*_{ij}, say to (P_2') with the corresponding cost $C^* < C^0$. Let $\{x^*_{ij}\}$ be the basic feasible solution to (P_1') corresponding to the cfs $\{y^*_{ij}\}$. Optimal objective value of (P_1') at an optimal solution $\{x^*_{ij}\}$ is equal to $C^* < C^0$, which contradicts the fact that $\{x^0_{ij}\}$ is the optimal solution to (P_1'). Hence, $\{y^0_{ij}\}$ must be an optimal cfs to (P_2').

Similarly, an optimal cfs to (P_2') will provide an optimal solution to (P_1').

3 Algorithm

Step 1: Given the fixed charge bi-criterion quadratic transportation problem. Separate it into two problems (P_1') and (P_1''). Let the flow be restricted to P. Introduce an additional column and an additional row with demand = $\sum_{i\in I} a_i - P$ and availability = $\sum_{j\in J} b_j - P$. Form the problem (P_2'). Find its initial basic feasible solution $\{y_{ij}'\}$. Let B be its corresponding basis.

Step 2: (a) Calculate the fixed cost of the current basic feasible solution and denote it by F^1 (current), where F^1(current) = $\sum_{i\in I} F_i$

(b) Find R_{ij} $\forall$ $(i, j) \notin B$
where

$$R_{ij} = \theta_{ij}(z_{ij}' - d_{ij})(z_{ij} - c_{ij}) - Z_1(z_{ij}' - d_{ij}) - Z_2(z_{ij} - c_{ij})$$

and denote it by $(R_{ij})_1$

Step 3: (a) Find $A_{ij}^1 = (R_{ij})_1 \times (E_{ij})_1$
where A_{ij}^1 is the change in the variable cost obtained on introducing a non basic cell (i, j) with value $(E_{ij})_1$ (for all $(i,j) \notin B$) into the basis.

(b) Find F_{ij}^1 (Difference) = Change in fixed cost = F_{ij}^1 (*NB*)- F^1 (*current*)
where F_{ij}^1 (*NB*) is the total fixed cost involved on introducing the variable x_{ij} with value $(E_{ij})_1$ (for all $(i, j) \notin B$) into the current basis to form a new basis.

(c) Find $\Delta_{ij}^1 = F_{ij}^1$ (Difference) + A_{ij}^1 $\forall$ $(i, j) \notin B$

If all $\Delta_{ij}^1 \geq 0$, then it is not possible to decrease the total cost *i.e*., (*variable cost* + *fixed cost*). Go to step 4 but if $\exists$ at least one $\Delta_{ij}^1 < 0$, find $\min\{\Delta_{ij}^1 / \Delta_{ij}^1 < 0, (i, j) \notin B\} = \Delta_{pq}$ (say). Then the cell (p, q) enters the basis.

Step 4: Let Z^1 be the optimal cost of (P_2') yielded by the basic feasible solution $\{y_{ij}^1\}$. Find
$T^1 = \underset{\substack{i\in I' \\ j\in J'}}{Max}\{t_{ij} \mid y_{ij}^1 > 0\}$ from the problem (P_2'')

Then the corresponding pair (Z^1, T^1) is the first cost- time trade off pair for the problem (P_2) and subsequently for the problem (P_1). To find the next best cost-time trade off pair, go to step 5.

Step 5: Define $c_{ij}^1 = \begin{cases} M & \text{if } t_{ij} \geq T^1 \\ c_{ij} & \text{if } t_{ij} < T^1 \end{cases}$

where M is a sufficiently large positive number. Form the corresponding fixed charge quadratic transportation problem with variable cost c_{ij}^1. Repeat the above process till we get an infeasible solution.

The complete set of cost-time trade off pairs of (P_1) at the end of qth iteration are given by
$(Z^1, T^1), (Z^2, T^2), \cdots, (Z^q, T^q)$
where $Z^1 < Z^2 < \cdots < Z^q$
and $T^1 > T^2 > \cdots > T^q$

Remark 4. The pair (Z^1, T^q) with minimum cost and minimum time is the ideal pair, which cannot be achieved in practice except in some trivial case.

Remark 5. The choice of c_{ij}^1 in step 5 will ensure the infeasibility of the basic feasible solution after a finite number of iterations.

4 Numerical Example

Fixed Charge Bi-criterion Quadratic Transportation Problem with restricted flow is:

$$\text{Minimize } \left\{ \left(\sum_{i\in I} \sum_{j\in J} c_{ij} x_{ij} \right) \left(\sum_{i\in I} \sum_{j\in J} d_{ij} x_{ij} \right) + \sum_{i\in I} F_i ,\ \underset{\substack{i\in I \\ j\in J}}{\text{Max}} \left\lfloor t_{ij} \mid x_{ij} > 0 \right\rfloor \right\}$$

$$\text{Subject to } \sum_{j\in J} x_{ij} \le a_i , \qquad i \in I$$

$$\sum_{i\in I} x_{ij} \le b_j , \qquad j \in J$$

$$\sum_{i\in I} \sum_{j\in J} x_{ij} = P(< \min(\sum_{i\in I} a_i , \sum_{j\in J} b_j))$$

$$x_{ij} \ge 0, \qquad i \in I,\ j \in J$$

Table-I gives the values of variable costs c_{ij}*'s* and d_{ij}*'s* (i = 1, 2, 3; j = 1, 2, 3) and Table-II gives the values of time t_{ij} (i = 1, 2, 3; j = 1, 2, 3)

Table-I

c_{11} →

d_{11} →

				a_i ↓
	5 4	9 2	9 1	19
	4 3	6 7	2 4	10
	2 2	1 9	1 4	11
b_j→	5	8	15	

Table-II

t_{11} →

12	8	2
10	13	11
6	9	14

The fixed costs are
F_{11} = 100, F_{12} = 50, F_{13} = 50, F_{21} = 150, F_{22} = 50,
F_{23} = 50, F_{31} = 200, F_{32} = 100, F_{33} = 50

The total cost, which is to be minimized, is given by $\left(\sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}x_{ij}\right)\left(\sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}x_{ij}\right)+\sum_{i=1}^{3} F_i$ where

$$F_i = \sum_{1=1}^{3} \delta_{i1} F_{i1} \text{ for } i = 1,2,3$$

where

$$\begin{aligned}
\delta_{i1} &= 1 \qquad \text{if } \sum_{j=1}^{3} x_{ij} > 0 \quad \text{for } i = 1,2,3 \\
&= 0 \qquad \text{otherwise} \\
\delta_{i2} &= 1 \qquad \text{if } \sum_{j=1}^{3} x_{il} > 7 \quad \text{for } i = 1,2,3 \\
&= 0 \qquad \text{otherwise} \\
\delta_{i3} &= 1 \qquad \text{if } \sum_{j=1}^{3} x_{il} > 10 \quad \text{for } i = 1,2,3 \\
&= 0 \qquad \text{otherwise}
\end{aligned}$$

Let the flow be restricted to $P = 25$ where $P = 25 < \min\left(\sum_{i=1}^{3} a_i = 40, \sum_{j=1}^{3} b_j = 28\right)$

Introducing a dummy source and a dummy destination in Table-I with

$$\begin{aligned}
&c_{i4} = 0 = d_{i4,} \quad i = 1,2,3 \\
&c_{4j} = 0 = d_{4j,} \quad j = 1,2,3 \\
&c_{44} = M = d_{44} \qquad \text{where } M \text{ is a large positive number.}
\end{aligned}$$

and

$$b_4 = \sum_{i=1}^{3} a_i - P = 40 - 25 = 15$$

$$a_4 = \sum_{j=1}^{3} b_j - P = 28 - 25 = 3$$

We form the corresponding problem (P_2'). Similarly, on introducing a dummy source and a dummy destination in Table-II with

$$\begin{aligned}
&\mathrm{t}_{i4} = 0 \ , \quad \mathrm{i} = 1,2,3 \\
&\mathrm{t}_{4j} = 0 \ , \quad \mathrm{j} = 1,2,3 \\
&\mathrm{t}_{44} > \underset{\substack{i\in I \\ j\in J}}{Max}\ \mathrm{t}_{ij} = 15
\end{aligned}$$

and taking $t_{44} = 18$ with $a_4 = 3$, $b_4 = 15$ we form the corresponding problem (P_2'').

Solve problem (P_2''). Its initial basic feasible solution by Vogel's Approximation method is given in Table-III.

Table-III

				F^1(current)↓
5 (4) 4	9 2	9 1	0 (15) 0	100
4 3	6 7	2 (10) 4	0 0	200
2 2	1 (6) 9	1 (5) 4	0 0	350
0 (1) 0	0 (2) 0	0 0	M M	0
				650

Here, $Z_1 = 51$, $Z_2 = 130$

Applying Steps 2(*b*) & 3, we find $R_{ij}^1, A_{ij}^1, \Delta_{ij}^1 \ \forall\ (i,j) \notin B$ which are given in Table- IV.

Table-IV

(i,j)	(1,2)	(1,3)	(2,1)	(2,2)	(2,4)	(3,1)	(3,4)	(4,3)
R_{ij}^1	402	638	-58	370	120	-234	245	255
E_{ij}^1	2	2	1	6	1	1	1	2
A_{ij}^1	804	1276	-58	2220	120	-234	245	510
F_{ij}^1 (NB)	650	650	650	650	650	650	600	650
F_{ij}^1 (Diff.)	0	0	0	0	0	0	-50	0
Δ_{ij}^1	804	1276	-58	2220	120	-234	195	510

Now, $\min\{\Delta_{ij}^1 / \Delta_{ij}^1 < 0, (i,j) \notin B\} = \min\{-58, -234\} = -234$

$\therefore$ cell (3, 1) enters the basis. We find the new solution.

Repeat the process till all $\Delta_{ij}^1 \geq 0$. The optimal solution is given in Table-V.

Table-V

				F^1(current)↓
5 4	9 (4) 2	9 7	0 (15) 0	100
4 3	6 7	2 (10) 4	0 0	200
2 (5) 2	1 (1) 9	1 (5) 4	0 0	350
0 0	0 (3) 0	0 0	M M	0
				650

Here $Z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} = 72, \quad Z_2 = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}x_{ij} = 87$

Table-VI

(i,j)	(1,1)	(1,3)	(2,1)	(2,2)	(2,4)	(3,4)	(4,1)	(4,3)
R_{ij}^1	33	288	164	196	56	136	396	360
E_{ij}^1	4	4	5	1	1	1	3	3
A_{ij}^1	132	1152	820	196	56	136	1188	1080
F_{ij}^1 (NB)	650	650	650	650	650	600	650	650
F_{ij}^1 (Diff.)	0	0	0	0	0	-50	0	0
Δ_{ij}^1	132	1152	820	196	56	86	1188	1080

Here $\Delta_{ij}^1 \geq 0 \quad \forall (i,j) \notin B$

It is now not possible to decrease the total cost (variable cost + fixed cost),

minimum cost $Z^1 = (72 \times 87) + 650$

$= 6264 + 650 = 6914$

and corresponding time = $T^1 = 14$.

Hence the first cost time trade off pair is (6914, 14).

Define $c_{ij}^1 = \begin{cases} M & \text{if } t_{ij} \geq T^1 = 14 \\ c_{ij} & \text{if } t_{ij} < T^1 = 14 \end{cases}$

On solving the new problem we obtain the second trade off pair as $(Z^2, T^2) = (7504, 11)$

Define $c_{ij}^2 = \begin{cases} M & \text{if } t_{ij} \geq T^2 = 11 \\ c_{ij} & \text{if } t_{ij} < T^2 = 11 \end{cases}$

we get the third cost time trade off pair as $(Z^3, T^3) = (8044, 9)$

Next, define $c_{ij}^3 = \begin{cases} M & \text{if } t_{ij} \geq T^3 = 9 \\ c_{ij} & \text{if } t_{ij} < T^3 = 9 \end{cases}$

and on solving we find that the problem is infeasible. Hence, the cost time trade off pairs are (6914, 14), (7504, 11) and (8044, 9).

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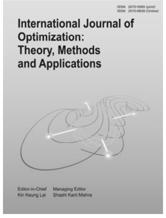

Generalizations of Variational Inequalities

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Abstract. A substantial number of papers have appeared recently proposing and analyzing generalizations of the variational inequality problem. In this paper, we present a non-exhaustive study about the main themes behind these generalizations.

Keywords: variational inequalities.

1 Introduction

The variational inequality problem, which was first introduced by Stampacchia [82] and Fichera [22] in 1964 independently in different contexts, has established itself as an important branch of applied mathematics, with a wide range of applications in numerous fields. Some excellent surveys of the variational inequality problem are available in [2,29,47,71]. A significant body of research is devoted to the development of algorithms for solving variational inequalities. Methods for solving variational inequalities include fixed point methods ([7,8,31,64]), projection methods ([69,79,80]), normal map (or Wiener-Hopf) equations ([44,45,74,77]), the auxiliary principle ([10,24,25,46,70,90]), descent and Newton-type methods ([23]), proximal-like methods ([4,30,71]), extra-gradient-type methods [61,87] and decomposition techniques [5,27,28,30,32,81,84,85]).

In the 70s and 80s, several important generalizations of the variational inequality problem were introduced: the quasi-variational inequality problem [6,7,46,50,51], the generalized inequality prob-

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lem [21,71], the generalized equations [75]. Of interest here is that both the generalized variational inequality and the generalized equation can be solved using the same algorithms, fundamentally, as the variation inequality problem. Thus, there is significant value in studying these problem classes in the unified framework of the generalized equation. In contrast, the quasi-variational inequality requires some fundamentally different ideas to reach to a solution.

In the last decade, a flurry of research has been devoted to studying further generalizations of variational inequalities. This work has resulted in no less than new problem classes. However, these different problem classes are variations on a small number of themes.

In this paper, we discuss the themes behind the various generalizations of variational inequalities.

We start with some background material, reviewing some of the classical literature of the field, and defining some notation conventions. Section 3, discusses the main ideas behind the various generalizations.

2 Background

2.1 Variational Inequalities

Let H be a Hilbert space with inner product $\langle .,. \rangle$ and norm $\| . \|$, let $f : H \to H$ be a single-valued operator and let $C \subset H$. The variational inequality problem $VI(C, f)$ is to find $x \in C$ such that

$$\langle f(x), y - x \rangle \geq 0, \quad \text{for all } y \in C. \tag{1}$$

It is usual to assume that C is closed and convex.

2.2 Generalized Equations

Let H be as above, and let T be a maximal monotone operator on H. (Recall that a monotone operator is a point to set mapping that maps to each point $x \in H$ a subset $T(x) \subset H$ in a such way that for every $x^1, x^2 \in H, y^1 \in T(x^1), y^2 \in T(x^2)$

$$\langle x^2 - x^1, y^2 - y^1 \rangle \geq 0.$$

A monotone operator is maximal if its graph♠ is not properly contained in the graph of any other monotone operator.) The notation $T : H \rightrightarrows H$ will be used to denote that T is a maximal monotone operator on H.

The generalized equation $GE(f, T)$, which was introduced by Robinson in [75] is to find $x \in H$ such that

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♠ $Graph(T) = \{(x, y) \in H \times H : y \in T(x)\}$.

$$0 \in f(x) + T(x).$$

To discuss the relationship between generalized equations and variational inequalities, it is necessary to define the *normal cone operator* N_C to a closed convex set *C*, which is defined by

$$N_C(x) = \begin{cases} \{z : \langle y - x, z \rangle \le 0, \forall y \in C\} & \textit{if } x \in C \\ \emptyset & \textit{otherwise.} \end{cases} \tag{2}$$

We note also that $N_C = \partial \delta_C$,where ∂ represents the sub-differential and $\partial \delta_C : H \to \bar{\Re}$ is the indicator function of *C*, defined by

$$\partial \delta_C(x) = \begin{cases} 0 & \textit{if } x \in C \\ +\infty & \textit{otherwise.} \end{cases}$$

Choosing $T = N_C$, the generalized equation *GE*(*f*, *T*) is equivalent to the variational inequality *VI*(*f*, *C*) [72]. However, using other choices of *T*, it is possible to formulate generalized equations that are not equivalent to any variational inequality. For example, if *T* is the sub-differential of a proper, lower semi-continuous extended real-valued function $\theta : H \to \bar{\Re}$ and if *f* = 0, then the generalized equation is equivalent to the first optimality conditions for minimizing θ. Thus, the generalized equations is a more general problem class than variational inequalities.

Of Significance, however, is the fact that the generalized equations can, in principle, be solved using fundamentally the same algorithmic ideas as variational inequalities. The key to do this is the *resolvent* operator $J_T : H \to H$ defined by

$$J_T := (I + \rho T)^{-1}, \tag{3}$$

where *I* represents the identity mapping and ρ a positive real number. According to Minty [40], it is known that the resolvent operator J_T is a monotone non-expansive single-valued function mapping *H* into *H* if *T* is a maximal monotone operator. Moreover, $x \in H$ is a solution to $y \in T(x)$ if and only if

$$J_T(x + \rho y) = x. \tag{4}$$

In the case where $T = N_C, J_T(x)$ is actually the Euclidean projection of *x* onto *C*. The resolvent operator provides a means for reformulating the generalized equation into a single-valued equation. For example, *GE*(*f*, *T*) is equivalent to the equation

$$f(J_T(x)) + \rho^{-1}(x - J_T(x)) = 0.$$

This equation appears in the literature both as the Wiener-Hopf equations ([44,45,78]) and also as the normal map ([74]). Alternatively, using (4).

Observe that if $T = N_C$, the normal cone operator, then *GE*(*f*, *T*) is equivalent to *VI*(*f*, *C*).

We shall see in the sequel, that the generalized equation is sufficiently flexible to include several generalizations of the variational inequality that have been proposed in recent years, including generalized variational inequalities and mixed variational inequalities.

An important generalization of variational inequalities that is not a subclass of generalized equations is the quasi-variational inequality. Let $C(H)$ be the collection of all closed convex subsets of H, $f : H \to H$ be a single-valued operator, and let $X : H \rightrightarrows C(H)$ be a set-valued mapping, which assigns to each $z \in H$ a closed convex subset $X(z)$ of H. The quasi-variational inequality problem *QVI* (X, f) is to find $x \in X(x)$ such that

$$\langle f(x), y - x \rangle \geq 0, \quad \text{for all } y \in X(x).$$

These problems have been considered and studied by Bensoussan and Lions [7] in the impulse control. Observe that if $X(x)$ is a fixed set C for all x, then the problem reduces to the classical variational inequality problem $VI(f, C)$, which is to find $x \in C$ such that

$$\langle f(x), y - x \rangle \geq 0, \quad \text{for all } y \in X.$$

The *QVI* problem generalizes the classical *VI* problem by allowing the set to vary as a function of the independent variable.

The quasi-variational problem was first introduced in ([7]), has been extensively [6,12,14,38,41] and represents a natural framework for many applications [66,71].

2.3 Notational Conventions

The following notations conventions will be used for the remainder of this paper. In discussing mappings, we shall use lower case roman letters (e.g., f) to represent single-valued operators, upper case letters (e.g., T) to represent point-to-set mappings, and Greek letters (e.g., θ) to represent extended real valued functions. We shall further distinguish between point-to-set mappings by reserving the letters S through V for maximal operators.

3 Themes for Generalizing Problem Classes

In this work, we focus on eight main themes which account for a large number of generalizations of variational inequalities and quasi-variational inequalities.[ϒ]

3.1 Theme 1: Replacing f by a Set-valued Operator

One method of generalizing variational inequalities is to replace the single-valued operator f by a set-valued operator $F : H \to 2^H$. This results in the General Variational Inequality $GVI(X, F)$, introduced in [21], which is to find $x \in C$ and $w \in F(x)$ such that

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[ϒ] The reader may meet in literature other specific generalizations that are not cited in this paper.

$$\langle w, y-x \rangle \geq 0, \quad \text{for all } y \in C. \tag{5}$$

For iterative algorithms see [17,18] and references therein. The same idea was applied to quasi-variational inequalities in [9] to produce the Generalized Quasi-Variational Inequality Problem *GQVI*(*X*, *F*), which is to find $x \in X(x)$ and $w \in F(x)$ such that

$$\langle w, y-x \rangle \geq 0, \quad \text{for all } y \in X(x). \tag{6}$$

For iterative algorithms see [47] and references therein.

3.2 Theme 2: Replacing *x* by *g*(*x*)

Another method of generalization is to introduce an additional operator $g: H \to H$ and replace *x* by *g*(*x*) in two places, yielding the general variational inequality, introduced by Noor [65-67]: Find $x \in H$ such that $g(x) \in C$ and

$$\langle f(x), y-g(x) \rangle \geq 0, \quad \text{for all } y \in C. \tag{7}$$

Observe that if *g* is invertible, this problem is equivalent to the classical variational inequality problem $VI(C, \hat{f})$, where $\hat{f}: H \to H$ is defined by $\hat{f}(x) := f(g^{-1}(x))$. From a theoretical standpoint, therefore, it is only interesting to study the cases where *g* is not invertible.

Again, the same idea can be applied to the quasi-variational inequality yielding the Generalized Quasi-Variational Inequality *GenQVI*(*X*, *f*, *g*), introduced in [43], find $x \in H$, such that $g(x) \in X(x)$ and

$$\langle f(x), y-g(x) \rangle \geq 0, \quad \text{for all } y \in X(x). \tag{8}$$

3.3. Theme 3: Alternatives to the Indicator Function

An alternative formulation of the variational inequality problem is to find $x \in H$ such that

$$\langle f(x), y-x \rangle + \delta_C(y) - \delta_C(x) \geq 0, \quad \text{for all } y \in H. \tag{9}$$

Observe that this formulation does not explicitly require *x* or *y* to be in the set *C*; instead this is enforced by the presence of the indicator function in 9). Indeed, if $x \notin C$, then the inequality will be violated by any $y \in C$. Furthermore, for any $y \notin C$, the left hand side will be $\{+\infty\}$ for all $x \in C$, so the inequality is automatically satisfied.

This formulation of the variational inequality problem suggests a method of generalization by replacing the indicator function δ_C by a lower-semi-continuous proper convex function $\phi: H \to \Re \cup \{+\infty\}$. This yields the mixed variational inequality $MVI(\phi, f)$: find $x \in H$ such that

$$\langle f(x), y-x \rangle + \phi(y) - \phi(x) \geq 0, \quad \text{for all } y \in H, \tag{10}$$

a problem studied by Hassouni and Moudafi [26] and Noor [52,53] using the resolvent operator and resolvent equation techniques and recently by Salmon *et al.* in [76]. For formulation and numerical techniques see [76].

We observe now that the restrictions imposed on the function ϕ ensure that $\partial\phi$ is a maximal monotone operator. The following proposition shows that the mixed variational inequality is a special case of the generalized equation. While the proof is straightforward, we could not find it in the literature, so we include it here.

Proposition 1: $MVI(\phi, f)$ is equivalent to the generalized equation $GE(\partial\phi, f)$.

Proof: We have that $VI(f,C)$ is equivalent to $MVI(\delta_C, f)$ (see Robinson [75]) and $VI(f,C)$ is equivalent to the $GE(\partial\delta_C, f)$

$$0 \in f(x) + N_C(x), \tag{11}$$

so

$$MVI(\delta_C, f) \equiv GE(\partial\delta_C, f). \tag{12}$$

Now, if we replace δ_C in $MVI(\delta_C, f)$ by a lower semi-continuous proper convex function $\phi : H \to \Re \bigcup \{+\infty\}$, then we have $MVI(\phi, f)$ (10), and if we replace N_C in (11) by $\partial\phi$ a maximal monotone operator, we will have $GE(\partial\phi, f) : 0 \in f(x) + \partial\phi(x)$, so from (12) we have that $MVI(\phi, f)$ is equivalent to $GE(\partial\phi, f)$. ◊

A natural extension of this idea to quasi-variational inequalities is to introduce a function $\psi : H \times H \to \Re \bigcup \{+\infty\}$ such that $\psi(.,y)$ is a lower-semi-continuous proper convex function for each fixed $y \in H$. The mixed quasi-variational inequality $MQVI(\psi, f)$ [71] is to find $x \in H$ such that

$$\langle f(x), y-x \rangle + \psi(y,x) - \psi(x,x) \geq 0, \quad \text{for all } y \in H. \tag{13}$$

Observe that if ψ is defined by $\psi(w,x) := \delta_X(x)(w)$, then $MQVI(\psi, f)$ reduces to the quasi-variational inequality.

3.4 Theme 4: Replacing $f(x)$ by $N(w,v)$

Let N an operator $N : H \times H \rightrightarrows H$, and $T, V : H \rightrightarrows C(H)$ be the multi-valued operators, where $C(H)$ is a family of all nonempty compact subsets of H. So we have the problem of finding $x \in H$ such that $w \in T(x)$, $v \in V(x)$ and

$$\langle N(w,v), y-x \rangle \geq 0, \quad \text{for all } y \in H \tag{14}$$

a problem considered and studied by Noor ([58], 1998).

3.5 Theme 5: Alternatives to the Inner-product

Replace $\langle f(x), y-x \rangle$ by $N(w,v)$ and $\partial\delta(x)$ by $A(x)$ a maximal monotone operator, so we have the problem of finding $x \in H, w \in f(x), v \in V(x)$ such that

$$0 \in N(w,v) + A(x), \tag{15}$$

which is called the set-valued variational inclusions introduced by Noor [59],[ξ] An extension to quasi-variational inclusions is to let $A(.,.): H \times H \to 2^H$ be a maximal monotone operator with respect to the first argument. The multi-valued quasi-variational inclusion problem $MQVI(F, V, A, N)$ is to find $x \in H, w \in F(x), v \in V(x)$ such that

$$0 \in N(w,v) + A(x,x). \tag{16}$$

3.6 Theme 6: Replacing x in $A(.,x)$ by a Set-valued Operator

Let $G: H \to C(H)$ be the multi-valued operator and $A(.,.): H \times H \to C(H)$ be a set-valued mapping such that for fixed $t \in H, A(.,t): H \to C(H)$ is a maximal monotone operator with respect to the first argument. For a given nonlinear single valued operator $N(.,.): H \times H \to H$, consider the problem of finding $x \in H, w \in F(x), v \in V(x), z \in G(x)$ such that

$$0 \in N(w,v) + A(x,z). \tag{17}$$

which is called generalized nonlinear set-valued mixed quasi-variational inequality and was introduced and studied in [33]. For more details see [1],

3.7 Theme 7: Replacing $(y-x)$ in VI(f,C) by an Operator $\eta(y,x)$

Another direction of generalization of $VI(f,C)$ is to replace $(y-x)$ by an operator $\eta(y,x)$, where $\eta: H \times H \to H$. Then, we have

$$\langle f(x), \eta(y,x) \rangle \geq 0, \quad \text{for all } y \in C, \tag{18}$$

which is called strongly nonlinear variational-like inequality and was introduced and studied by Noor [55,56], and has many applications in optimization.

3.8 Combinations of Themes

Various combinations of the themes described above have been considered to produce a large collection of generalizations of variational and quasi-variational inequalities. The following is a quick summary of what has appeared in the literature:

Themes 1 and 2. Combining themes 1 and 2 yields the Generalized Nonlinear Variational Inequality $GNVI(C, F, g)$ [50], which is to find $x \in H$ and $w \in F(x)$ such that $g(x) \in C$ and

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[ξ] See [42,75,86] for related works.

$$\langle w, y-g(x)\rangle \geq 0, \quad \text{for all } y \in C.$$

Applying these themes to quasi-variational inequality yields the Generalized Nonlinear Quasi-Variational Inequality $GNQVI(X, F, g)$ [71] which is to find $x \in H$ and $w \in F(x)$ such that $g(x) \in X(x)$ and

$$\langle w, y-g(x)\rangle \geq 0, \quad \text{for all } y \in X(x).$$

Themes 1 and 3. Combining themes 1 and 3 yields

$$\langle w, y-x\rangle + \phi(y) - \phi(x) \geq 0, \quad \text{for all } y \in H,$$

which is called the generalized mixed variational inequalities problem [54] and

$$\langle w, y-x\rangle + \psi(y, x) - \psi(x, x) \geq 0, \quad \text{for all } y \in H,$$

a problem studied by Noor [63] using the auxiliary principle techniques.

Themes 1,2 and 3. Combining themes 1, 2 and 3 yields

$$\langle w, y-g(x)\rangle + \phi(y) - \phi(g(x)) \geq 0, \quad \text{for all } y \in H,$$

a problem studied by Noor [71] and

$$\langle w, y-g(x)\rangle + \psi(y, g(x)) - \psi(x, g(x)) \geq 0, \quad \text{for all } y \in H,$$

which is called the set-valued mixed quasi-variational inequality considered by Noor [59].

Themes 2 and 3. Combining themes 2 and 3 yields:

$$\langle f(x), y-g(x)\rangle + \phi(y) - \phi(g(x)) \geq 0, \quad \text{for all } y \in H,$$

a problem studied by Noor [71] and

$$\langle f(x), g(y)-g(x)\rangle + \phi(g(y)) - \phi(g(x)) \geq 0, \quad \text{for all } g(y) \in H,$$

which is called the general mixed variational inequality (see Noor [62]).

Themes 2 and 4. Combining themes 2 and 4 yields:

$$\langle N(w, v), y-g(x)\rangle \geq 0, \quad \text{for all } y \in C,$$

a problem considered and studied by Noor [44] (Using the projection method and the wiener-Hopf equations technique.)

Themes 3 and 4. Combining themes 3 and 4 yields:

$$\langle N(w, v), y-x\rangle + \phi(y) - \phi(x) \geq 0, \quad \text{for all } y \in H,$$

which was studied by Noor [63] (Using the auxiliary principle and resolvent techniques.)

Themes 2, 3 and 4. Combining themes 2, 3 and 4 yields:

$$\left\langle N(w,v), y-g(x)\right\rangle + \phi(y) - \phi(g(x)) \geq 0, \quad \text{for all } y \in H,$$

a problem called the generalized mixed multi-valued variational inequality and introduced by Noor *et al.* [60]. See also [1,33,35].

Themes 2 and 5. Combining themes 2 and 5 yields:

$$0 \in N(w,v) + A(g(x)),$$

which is called the generalized set-valued quasi-variational inclusions and

$$0 \in N(w,v) + A(g(x),x),$$

which is called the multi-valued mixed-quasi-variational inclusions $MQVI(F,V,g,A,N)$ and was studied by Noor [66].

Themes 2 and 6. Combining themes 2 and 6 yields:

$$0 \in N(w,v) + A(g(x),z),$$

a problem called the generalized multi-valued mixed quasi variational inequality problem *GMQVI* (*F*, *V*, *G*, *g*, *A*, *N*) and studied by Huang *et al.* [33] and recently by Al-Shemas and Billups [1].

Themes 4 and 7. Combining themes 4 and 7 yields:

$$\left\langle N(w,v), \eta(y,x)\right\rangle \geq 0, \text{ for all } y \in H, \tag{19}$$

which is the generalized variational-like inequality $GV-LI(N,F,V,\eta,C)$ [56]. It has been shown in [72] that the nonconvex non-monotone and multi-valued problem arising in structural analysis can be formulated in terms of generalized variational-like inequalities (19). For further applications of the problem (19) see Noor [49,55,56]. Parida and Sen [73], Tian [83] and Cubiotti [13] have shown that many problems arising in optimization and economics can be studied by the generalized variational inequalities of the the type (19).

The problem (19) can be extended for generalized mixed variational-like inequalities *GMV-LI* (N, F, V, η, b) of the types:

$$\left\langle N(w,v), \eta(y,x)\right\rangle + b(x,y) - b(x,x) \geq 0, \quad \text{for all } x \in H \tag{20}$$

where the form $b(.,.) : H \times H \rightarrow \Re$ which is non-differentiable and satisfies the properties:

1. $b(.,.)$ is linear in the first argument,
2. $b(.,.)$ is bounded, that is, there exists a constant $\nu > 0$ such that
3. $b(x,y) \leq \nu \| x \| \| y \|,$ for all $x, y \in H$,
4. $b(x,y) - b(x,z) \leq b(x, y-z),$ for all$x, y, z \in H$.

Problem 20) was studied by Noor [55] in 1997, Huang and Deng [36] in 2001 and recently in 2005 by Zeng *et al.*[90].

Conclusion

In this work, we expose a certain number of themes gathering the main ideas behind a large number of generalizations of variational inequality models. As it was said in the introduction, this subject is wide enough, and may be other directions of generalizations are missed in this work. For a part of completeness, we finish this paper by pointing some other generalizations:

- *Generalized quasi-variational-like inclusion problem with Fuzzy mappings* (see [37,49,55,56, 90] and references therein). This model can be obtained from theme 2-6 using fuzzy set-valued mappings.
- *Generalized vector quasi-variational-like inequality problem* (see [3,19] and references therein).

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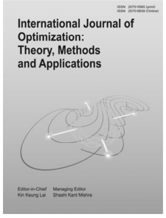

Note On $(F,\alpha,\rho,d)-V$ -type I Functions

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Abstract. In this paper we introduce the concept of partial $(F,\alpha,\rho,d)-V$ -type I function and its generalizations which are then applied to establish sufficient optimality conditions and duality results for a general minimax programming problem. A Vector-valued partial Lagrangian is also introduced. Mixed saddle points are related to the optimal solutions of the general minimax programming problem.

Keywords: minimax programming problem, generalized type I function, sufficiency, duality, partial Lagrangian, mixed saddle point.

1 Introduction

Minmax theory treats a class of mathematical programming problems which involve not only minimization or maximization but also a combination of both. The field of minimax programming problem has grown remarkably in many important areas like game theory, chebychev approximation, economics and financial planning. The practical application of minmax model usually involves discrete objects such as Ballastic missiles, Facility location problem etc. Thus, the discrete formulation of the problem seems more appropriate. The significance of minmax model is well known in many areas of decision-making, engineering design and mathematical programming.

Various classes of functions have been defined for the purpose weakening the limitation of convexity in mathematical programming. Hanson [3] introduced the class of invex function as a broad

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generalization of convexity. Later, Hanson and Mond [4] introduced two new classes of functions called type I and type II functions. Preda [10] introduced the notion of (F,ρ) -convexity. Hachimi and Agheezaf [6] introduced (F,α,ρ,d) -type I functions. Later, Gulati et.al [2] generalized these functions to $(F,\alpha,\rho,d)-V$ -type I functions.

In this paper, we move a step further in this direction and relax the notion of $(F,\alpha,\rho,d)-V$ - type I functions to the concept of partial $(F,\alpha,\rho,d)-V$ -type I function and its generalizations. We use this relaxed notion to establish sufficient optimality conditions and duality results for a general minimax programming problem. Mixed type duality is introduced for equivalent optimization problem. Weak and strong duality results are established for mixed type dual. A vector-valued partial Lagrangian is also introduced. Mixed saddle point for the same is shown to be optimal for the corresponding Minimax programming problem.

2 Preliminaries and Definitions

We concentrate on the following general minimax programming problem in which several objectives are to be optimized simultaneously and overall objective is to minimize the largest of these objectives.

$$\text{(P)} \qquad \underset{x\in X}{\text{Minimize}}\ \ \underset{1\le i\le k}{\text{Maximize}}\ (f_i(x))$$

$$\text{subject to} \quad g_j(x)\le 0, \qquad j=1,\cdots,m,$$

where X is an open subset of R^n. The functions $f=(f_1,\cdots,f_k)$ and $g=(g_1,\cdots,g_m)$ are real valued differentiable functions on X and $S=\{x\in X: g_j(x)\le 0,\ j=1,\cdots,m\}$ is non-empty and compact.

Let F be a sublinear functional and let $d: X\times X\to R$ be a metric. Let $K=\{1,\cdots,k\}$ and $M=\{1,\cdots,m\}$ be the index sets. Let $P\subseteq K$ and $Q\subseteq M$ be the subsets of K and M respectively where $P=\{1,\cdots,p\}$ and $Q=\{1,\cdots,q\}$.

Definition 1. $(f_i,g_j), i=1,\cdots,p$ and $j=1,\cdots,q$, is said to be partially $(F,\alpha,\rho,d)-V$ -type I at $x^*\in X$, if there exist vector functions $\alpha(x,x^*)=(\alpha_1^1(x,x^*),\cdots,\alpha_p^1(x,x^*),\alpha_1^2(x,x^*),\cdots,\alpha_q^2(x,x^*))$ and $\rho=(\rho_1^1,\cdots,\rho_p^1,\rho_1^2,\cdots,\rho_q^2,)$, with $\alpha_i^1(x,x^*),\alpha_j^2(x,x^*): X\times X\to R_+\setminus\{0\}$ and $\rho_i^1,\ \rho_j^2\in R$ for $i\in P, j\in Q$ such that for all $x\in X$,

$$f_i(x)-f_i(x^*)\geqq F(x,x^*;\alpha_i^1(x,x^*)\nabla f_i(x^*))+\rho_i^1 d^2(x,x^*)$$
$$-g_j(x^*)\geqq F(x,x^*;\alpha_j^2(x,x^*)\nabla g_j(x^*))+\rho_j^2 d^2(x,x^*).$$

If the first inequality is satisfied as

$$f_i(x)-f_i(x^*) > F(x,x^*;\alpha_i^1(x,x^*)\nabla f_i(x^*)) + \rho_i^1 d^2(x,x^*),$$

then (f_i,g_i) is said to be semistrictly partially $(F,\alpha,\rho,d)-V$ -type I at x^*.

Definition 2. $(f_i, g_i),\ i=1,\cdots,p$ and $j=1,\cdots,q$, is said to be partially quasi $(F,\bar{\alpha},\bar{\rho},d)-V$ type I at $x^*\in X$, if there exist vector functions $\bar{\alpha}(x,x^*)=(\bar{\alpha}_1^1(x,x^*),\cdots,\bar{\alpha}_p^1(x,x^*),\bar{\alpha}_1^2(x,x^*),\cdots,\bar{\alpha}_q^2(x,x^*))$ and $\bar{\rho}=(\bar{\rho}^1,\bar{\rho}^2)\in R^2$, with $\bar{\alpha}_i^1(x,x^*),\bar{\alpha}_j^2(x,x^*):X\times X\to R_+\setminus\{0\}$ for $i\in P, j\in Q$, such that for all $x\in X$,

$$\sum_{i=1}^{p}\bar{\alpha}_i^1(x,x^*)\,f_i(x)\leqq\sum_{i=1}^{p}\bar{\alpha}_i^1(x,x^*)\,f_i(x^*)\Rightarrow F(x,x^*;\sum_{i=1}^{p}\nabla f_i(x^*))\leqq-\bar{\rho}^1\,d^2(x,x^*),$$

$$-\sum_{j=1}^{q}\bar{\alpha}_j^2(x,x^*)\,g_j(x^*)\leqq 0\Rightarrow F(x,x^*;\sum_{j=1}^{q}\nabla g_j(x^*))\leqq-\bar{\rho}^2 d^2(x,x^*)$$

Definition 3. $(f_i,g_j),i=1,\cdots,p$ and $j=1,\cdots,q$, is said to be partially pseudo $(F,\widehat{\alpha},\widehat{\rho},d)-V$ -type-I at $x^*\in X$, if there exist vector functions $\widehat{\alpha}(x,x^*)=(\widehat{\alpha}_1^1(x,x^*),\cdots,\widehat{\alpha}_p^1(x,x^*),\widehat{\alpha}_1^2(x,x^*),\cdots,\widehat{\alpha}_q^2(x,$ $x^*))$ and $\widehat{\rho}=(\widehat{\rho}^1,\widehat{\rho}^2)\in R^2$, with $\widehat{\alpha}_i^1(x,x^*),\widehat{\alpha}_j^2(x,x^*):X\times X\to R_+\setminus\{0\}$ for $i\in P, j\in Q$, such that for all $x\in X$,

$$F(x,x^*;\sum_{i=1}^{p}\nabla f_i(x^*))\geqq-\widehat{\rho}^1 d^2(x,x^*)\Rightarrow\sum_{i=1}^{p}\widehat{\alpha}_i^1(x,x^*)f_i(x)\geqq\sum_{i=1}^{p}\widehat{\alpha}_i^1(x,x^*)f_i(x^*),$$

$$F(x,x^*;\sum_{j=1}^{q}\nabla g_j(x^*))\geqq-\widehat{\rho}^2 d^2(x,x^*)\Rightarrow-\sum_{j=1}^{q}\widehat{\alpha}_j^2(x,x^*)\,g_j(x^*)\geqq 0,$$

Definition 4. $(f_i,g_j),i=1,\cdots,p$ and $j=1,\cdots,q$, is said to be partially pseudoquasi $(F,\tilde{\alpha},\tilde{\rho},d)-V$ -type I at $x^*\in X$, if there exist vector functions $\tilde{\alpha}(x,x^*)=(\tilde{\alpha}_1^1(x,x^*),\cdots,\tilde{\alpha}_p^1(x,x^*),\tilde{\alpha}_1^2(x,x^*),\cdots,\tilde{\alpha}_q^2$ $(x,x^*))$ and $\tilde{\rho}=(\tilde{\rho}^1,\tilde{\rho}^2)\in R^2$, with $\tilde{\alpha}_i^1(x,x^*),\tilde{\alpha}_j^2(x,x^*):X\times X\to R_+\setminus\{0\}$ for $i\in P, j\in Q$, such that for all $x\in X$,

$$\sum_{i=1}^{p}\tilde{\alpha}_i^1(x,x^*)f_i(x)<\sum_{i=1}^{p}\tilde{\alpha}_i^1(x,x^*)f_i(x^*)\Rightarrow F(x,x^*;\sum_{i=1}^{p}\nabla f_i(x^*))<-\tilde{\rho}^1 d^2(x,x^*),$$

$$-\sum_{j=1}^{q}\tilde{\alpha}_j^2(x,x^*)g_j(x^*)\leqq 0\Rightarrow F(x,x^*;\sum_{j=1}^{q}\nabla g_j(x^*))\leqq-\tilde{\rho}^2 d^2(x,x^*).$$

If the first inequality is satisfied as

$$\sum_{i=1}^{p}\tilde{\alpha}_i^1(x,x^*)f_i(x)\ \leqq\ \sum_{i=1}^{p}\tilde{\alpha}_i^1(x,x^*)f_i(x^*)\ \Rightarrow\ F(x,x^*;\sum_{i=1}^{p}\nabla f_i(x^*))<-\tilde{\rho}^1 d^2(x,x^*),$$

then (f_i,g_j) is said to be strictly partially pseudoquasi $(F,\tilde{\alpha},\tilde{\rho},d)-V$ -type I at x^*.

Definition 5. $(f_i,g_j),i=1,\cdots,p$ and $j=1,\cdots,q$, is said to be partially quasipseudo $(F,\hat{\alpha},\hat{\rho},d)-V$ -type I at $x^*\in X$, if there exist vector functions $\hat{\alpha}(x,x^*)=(\hat{\alpha}_1^1(x,x^*),\cdots,\hat{\alpha}_p^1(x,x^*),\hat{\alpha}_1^2(x,x^*),\cdots,\hat{\alpha}_q^2$ $(x,x^*))$ and $\hat{\rho}=(\hat{\rho}^1,\hat{\rho}^2)\in R^2$, with $\hat{\alpha}_i^1(x,x^*),\hat{\alpha}_j^2(x,x^*):X\times X\to R_+\setminus\{0\}$ for $i\in P, j\in Q$, such that for all $x\in X$,

$$F(x,x^*;\sum_{i=1}^{p}\nabla f_i(x^*)) > -\hat{\rho}^1 d^2(x,x^*) \Rightarrow \sum_{i=1}^{p}\hat{\alpha}_i^1(x,x^*)f_i(x) > \sum_{i=1}^{p}\hat{\alpha}_i^1(x,x^*)f_i(x^*),$$

$$-\sum_{j=1}^{q}\hat{\alpha}_j^2(x,x^*)g_j(x^*)<0\Rightarrow F(x,x^*;\sum_{j=1}^{q}\nabla g_j(x^*)) < -\hat{\rho}^2 d^2(x,x^*),$$

if the second inequality is satisfied as

$$-\sum_{j=1}^{q}\hat{\alpha}_j^2(x,x^*)g_j(x^*)\leqq 0 \Rightarrow F(x,x^*;\sum_{j=1}^{q}\nabla g_j(x^*)) < -\hat{\rho}^2 d^2(x,x^*)$$

then (f_i, g_j) is said to be partially quasistrictly pseudo $(F,\hat{\alpha},\hat{\rho},d)-V$ -type I at x^*.

If the general minimax problem (P) reaches a finite optimal, then it may be equivalently expressed as

(EP) Minimize q

subject to $f_i(x)\le q, \quad i\in K$ (1)

$g_j(x)\le 0, \quad j\in M$ (2)

$q\in R, \quad x\in X$

Lemma 1. [1] If (x,q) is (EP)-feasible, then x is (P)-feasible. If x is (P)-feasible then there exists $q\in R$ such that (x,q) is (EP)-feasible.

Lemma 2. [1] x^* is (P)-optimal with the corresponding optimal value of the (P) objective equal to q^* if and only if (x^*, q^*) is (EP)-optimal with corresponding optimal value of (EP)-objective equal to q^*.

3 Optimality Conditions

In this section, we provide necessary and sufficient optimality conditions for minimax programming problem to possess an optimal solution.

Theorem 1. (Necessary Conditions). Let $x^*\in X$ be (P)-optimal with optimal objective value q^*. Let an appropriate constraint qualification [8] holds for (EP), then there exist $\lambda^*\in R_+^k, \mu^*\in R_+^m$ such that (x^*,λ^*,μ^*) satisfies

$$\sum_{i=1}^{k}\lambda_i^*\nabla f_i(x^*)+\sum_{j=1}^{m}\mu_j^*\nabla g_j(x^*)=0, \tag{3}$$

$$\lambda_i^*[f_i(x^*)-q^*]=0, \qquad i\in K \tag{4}$$

$$\mu_j^* g_j(x^*)=0, \qquad j\in M \tag{5}$$

$$f_i(x^*)-q^* \le 0, \qquad i\in K \tag{6}$$

$$g_j(x^*)\le 0, \qquad j\in M \tag{7}$$

$$\sum_{i=1}^{k}\lambda_i^* =1 \tag{8}$$

$$\lambda^* \geqq 0,\ \mu^* \geqq 0 \tag{9}$$

Proof: It follows directly by writing the necessary optimality conditions for the problem (EP).

Theorem 2. (Sufficient Conditions) Let x^* be (P)-feasible and let there exist scalars $\lambda_i^* \geqq 0,\ i\in K$ with $\sum_{i=1}^{k}\lambda_i^* =1$ and $\mu_j^* \geqq 0,\ j\in M$ such that (3) - (9) are satisfied. Further let,

(i) $(\lambda_i^* f_i, \mu_j^* g_j),\ i\in I(x^*)$ and $j\in J(x^*)$, be partially pseudoquasi $(F,\alpha,\rho,d)-V$ - type I at x^*.

(ii) $\rho^1+\rho^2>0$,

where $I(x^*)=\{i\in K: f_i(x^*)-q^*=0\}$ and $J(x^*)=\{j\in M: g_j(x^*)=0\}$.

Then x^* is (P)-optimal with the corresponding optimal objective value equal to q^*.

Proof. Let x^* be not an optimal solution for (P). Then from Lemma 2, it follows that (x^*,q^*) is not an optimal solution for (EP). Thus there exists (EP)-feasible (x,q) with $x\ne x^*$ and $q<q^*$

From (1) it follows that

$$f_i(x)\le q<q^*=f_i(x^*), \qquad i\in I(x^*)$$

i.e $f_i(x)<f_i(x^*), \qquad i\in I(x^*)$

From (4), it follows that $\lambda_i^*=0$ for each $i\notin I(x^*)$ and therefore from (8), $\sum_{i\in I(x^*)}\lambda_i=1$, which ensures the existence of atleast one $\lambda_i^*>0,\ i\in I(x^*)$ and since $\alpha_i^1(x,x^*)>0, i\in K,$ the above inequality implies that

$$\sum_{i\in I(x^*)}\alpha_i^1(x,x^*)\lambda_i^* f_i(x) < \sum_{i\in I(x^*)}\alpha_i^1(x,x^*)\lambda_i^* f_i(x^*) \tag{10}$$

Also since $g_j(x^*)=0,\ j\in J(x^*)$, thus, we get

$$-\sum_{j\in J(x^*)}\alpha_j^2(x,x^*)\ \mu_j^* g_j(x^*) = 0 \tag{11}$$

By hypothesis (*i*), (10) and (11) imply

$$F(x,x^*;\ \sum_{i\in I(x^*)}\lambda_i^*\nabla f_i(x^*)) < -\rho^1 d^2(x,x^*) \tag{12}$$

$$F(x,x^*;\ \sum_{j\in J(x^*)}\mu_j^*\nabla g_j(x^*)) \leqq -\rho^2 d^2(x,x^*) \tag{13}$$

Using $\lambda_i^* = 0,\ i \notin I(x^*)$ and $\mu_j^* = 0,\ j \notin J(x^*)$, (12) and (13) can be rewritten as

$$F(x,x^*;\sum_{i=1}^{k}\lambda_i^*\nabla f_i(x^*)) \ < -\rho^1 d^2(x,x^*) \tag{14}$$

$$F(x,x^*;\sum_{j=1}^{m}\mu_j^*\nabla g_j(x^*)) \leqq -\rho^2 d^2(x,x^*) \tag{15}$$

Now $0 = F(x,x^*;\sum_{i=1}^{k}\lambda_i^*\nabla f_i(x^*) + \sum_{j=1}^{m}\mu_j^*\nabla g_j(x^*))$

$\leqq F(x,x^*;\sum_{i=1}^{k}\lambda_i^*\nabla f_i(x^*)) + F(x,x^*;\sum_{j=1}^{m}\mu_j^*\nabla g_j(x^*))$ (∵ F is sublinear)

$< -(\rho^1+\rho^2)\, d^2(x,x^*)$ (using (14) and (15))

$< 0,$ (using hypothesis (*ii*))

which is not possible. Hence x^* is (P)-optimal.

4 Mixed Type Duality

The present section is devoted to develop the duality relationship between (P) and its mixed dual (D). Let I_1 be a subset of K and $I_2 = K / I_1$ and J_1 is a subset of M and $J_2 = M / J_1$. The mixed dual for (P) is given by

(D) Maximize v

$$\text{subject to } \sum_{i=1}^{k}\lambda_i\nabla[f_i(u) + \sum_{j=1}^{m}\mu_j g_j(u)] = 0 \tag{16}$$

$$\lambda_i[f_i(u) - v] + \sum_{j\in J_1}\mu_j g_j(u) \geqq 0, \qquad \forall i \in I_1 \tag{17}$$

$$\mu_j g_j(u) \geqq 0, \qquad \forall j \in J_2 \tag{18}$$

$$\lambda \in R_+^k\ ,\ \sum_{i=1}^{k}\lambda_i = 1, \tag{19}$$

$$\mu \in R_+^m, u \in X, v \in R$$

Theorem 3. (Weak Duality) Let x be (P)-feasible and (u,v,λ,μ) be (D)-feasible. Further, assume that

(i) $(\lambda_i f_i + \sum_{j\in J_1}\mu_j g_j, \mu_j g_j), i \in I_1$ and $j \in J_2$, be partially pseudoquasi $(F,\alpha,\rho,d)-V$ -type *I* at *u*.

(ii) $\rho^1 + \rho^2 > 0,$

where $I_1 = \{i \in K : \lambda_i > 0\}$.
Then, $q \geq v$.

Proof: Suppose $q < v$ (20)

Since x is (P)-feasible, therefore by Lemma 1 there exists $q \in R$ such that (x, q) is (EP)-feasible. Therefore, from (1), (2), (19) and (20), we have

$$\lambda_i f_i(x) + \sum_{j \in J_1} \mu_j g_j(x) \leq \lambda_i q < \lambda_i v, \qquad i \in I_1$$

Using (17) in the above inequality, we have

$$\lambda_i f_i(x) + \sum_{j \in J_1} \mu_j g_j(x) < \lambda_i f_i(u) + \sum_{j \in J_1} \mu_j g_j(u), \qquad i \in I_1$$

Since $\alpha_i^1(x,u) > 0$, for $i \in I_1$, therefore

$$\sum_{i \in I_1} \alpha_i^1(x,u)\{\lambda_i f_i(x) + \sum_{j \in J_1} \mu_j g_j(x)\} < \sum_{i \in I_1} \alpha_i^1(x,u)\{\lambda_i f_i(u) + \sum_{j \in J_1} \mu_j g_j(u)\} \tag{21}$$

Also for $j \in J_2$, $\alpha_j^2(x,u) > 0$, therefore from (18), we have

$$-\sum_{j \in J_2} \alpha_j^2(x,u) \mu_j g_j(u) \leqq 0 \tag{22}$$

Using hypothesis (*i*), (21) and (22) imply

$$F(x,u; \sum_{i \in I_1} \lambda_i \nabla f_i(u) + \sum_{j \in J_1} \mu_j \nabla g_j(u)) < -\rho^1 d^2(x,u) \tag{23}$$

$$F(x,u; \sum_{j \in J_2} \mu_j \nabla g_j(u)) \leqq -\rho^2 d^2(x,u) \tag{24}$$

Using $\lambda_i = 0, i \notin I_1$, (23) can be rewritten as

$$F(x,u; \sum_{i=1}^{k} \lambda_i \nabla f_i(u) + \sum_{j \in J_1} \mu_j \nabla g_j(u)) < -\rho^1 d^2(x,u) \tag{25}$$

Now

$$\begin{aligned}
0 &= F(x,u,0) \\
&= F(x,u; \sum_{i=1}^{k} \lambda_i \nabla f_i(u) + \sum_{j=1}^{m} \mu_j \nabla g_j(u)) && \text{(using (16))} \\
&= F(x,u; (\sum_{i=1}^{k} \lambda_i \nabla f_i(u) + \sum_{j \in J_1} \mu_j \nabla g_j(u)) + \sum_{j \in J_2} \mu_j \nabla g_j(u)) \\
&\leq F(x,u; \sum_{i=1}^{k} \lambda_i \nabla f_i(u) + \sum_{j \in J_1} \mu_j \nabla g_j(u)) + F(x,u; \sum_{j \in J_2} \mu_j \nabla g_j(u)) && \text{(F is sublinear)} \\
&< -(\rho^1 + \rho^2) d^2(x,u) && \text{(using (24) and (25))} \\
&< 0 && \text{(using assumption (ii))}
\end{aligned}$$

which is not possible. Hence $q \geq v$.

Now we introduce another dual (D') to the equivalent problem (EP) and establish weak duality relations between (EP) and (D'). Strong duality is also established.

(D') $\quad$ $\underset{v \in R}{\text{Maximize}}\ v$

$$\text{subject to } \sum_{i=1}^{k} \lambda_i \nabla f_i(u) + \sum_{j=1}^{m} \mu_j \nabla g_j(u) = 0 \tag{26}$$

$$\lambda_i [f_i(u) - v] = 0, \qquad i \in K \tag{27}$$

$$\mu_j\, g_j(u) = 0, \qquad j \in M \tag{28}$$

$$\sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geqq 0,\ i \in K,\ \mu_j \geqq 0,\ j \in M.$$

Theorem 4. (Weak Duality) Let x be (P)-feasible with the corresponding objective value equal to q and let (u, v, λ, μ) be (D')-feasible. Further, assume that

(i) $(\lambda_i f_i, \mu_j\, g_j)$, $i \in I_1$ and $j \in J_1$ be partially pseudoquasi $(F, \tilde{\alpha}, \tilde{\rho}, d) - V$ -type I at u.

(ii) $\tilde{\rho}^1 + \tilde{\rho}^2 > 0$

where $I_1 = \{i \in K : \lambda_i > 0\}$ and $J_1 = \{j \in M : \mu_j > 0\}$

Then, $q \geq v$.

Proof: Let us assume on the contrary that $q < v$.

Since x is (P)-feasible, therefore by Lemma 1 there exists $q \in R$ such that (x, q) is (EP)-feasible, therefore

$$f_i(x) \leq q < v, \qquad i \in K$$

For $i \in I_1$, it follows from (27), in view of the above inequality that

$$\lambda_i\, f_i(x) < \lambda_i\, f_i(u), \quad i \in I_1$$

Since $\tilde{\alpha}_i^1(x,u) > 0, i \in I_1$, we obtain

$$\sum_{i \in I_1} \tilde{\alpha}_i^1(x,u)\, \lambda_i\, f_i(x) < \sum_{i \in I_1} \tilde{\alpha}_i^1(x,u)\, \lambda_i\, f_i(u)$$

Also for $j \in J_1$, $\tilde{\alpha}_j^2(x,u) > 0$, therefore from (28), we have

$$-\sum_{j \in J_1} \tilde{\alpha}_j^2(x,u)\, \mu_j\, g_j(u) = 0$$

Using hypothesis (*i*), we get

$$F(x,u; \sum_{i \in I_1} \lambda_i \nabla f_i(u)) < -\tilde{\rho}^1\, d^2(x,u) \tag{29}$$

$$F(x,u;\sum_{j\in J_1}\mu_j\nabla g_j(u))\leqq-\tilde{\rho}^2\,d^2(x,u) \tag{30}$$

Again using $\lambda_i=0, i\notin I_1$ and $\mu_j=0,\ j\notin J_1$, we can rewrite (29) and (30) as

$$F(x,u;\sum_{i=1}^{k}\lambda_i\nabla f_i(u))<-\tilde{\rho}^1\,d^2(x,u) \tag{31}$$

$$F(x,u;\sum_{j=1}^{m}\mu_j\nabla g_j(u))\leqq-\tilde{\rho}^1 d^2(x,u) \tag{32}$$

Now

$$\begin{aligned}0&=F(x,u;\sum_{i=1}^{k}\lambda_i\nabla f_i(u)+\sum_{j=1}^{m}\mu_j\nabla g_j(u))\\&\leqq F(x,u;\sum_{i=1}^{k}\lambda_i\nabla f_i(u))+F(x,u;\sum_{j=1}^{m}\mu_j\nabla g_j(u)) &&\text{(F is sublinear)}\\&<-(\tilde{\rho}^1+\tilde{\rho}^2)\,d^2(x,u) &&\text{(by (31) and (32))}\\&<0 &&\text{(using assumption (ii))}\end{aligned}$$

which is not possible. Hence $q\geq v$.

Theorem 5. (Strong Duality) Let x^* be (P)-optimal and let an appropriate constraint qualification holds at x^* [8] for (EP). Then there exist $q^*\in R$, $\lambda^*\in R^k$, $\mu^*\in R^m$ such that $(x^*,q^*,\lambda^*,\mu^*)$ is (D) or (D')-feasible and the corresponding objective values of (EP) and (D) or (D') are equal. If also, the hypothesis of Theorem 3 or 4 hold, then $(x^*,q^*,\lambda^*,\mu^*)$ is (D) or (D')-optimal.

Proof: Since x^* is (P)-optimal, it follows from Lemma 2 that $\exists\ q^*\in R$ such that (x^*,q^*) is (EP) -optimal. Also an appropriate constraint qualification [8] holds at x^* therefore by Theorem 1, there exists $(\lambda^*,\mu^*)\in R_+^{k+m}$ such that conditions (3)-(9) are satisfied. Hence $(x^*,q^*,\lambda^*,\mu^*)$ is (D) or (D') -feasible with $v=q^*$, and objective values of (EP) and (D) or (D') are equal. If also, the hypotheses of Theorem 3 or 4 hold then $(x^*,q^*,\lambda^*,\mu^*)$ is (D) or (D') -optimal.

5. Partial Lagrangian and Mixed Saddle Point

Saddle point of the Lagrangian is always a global minimizer for the inequality constrained minimization problem. Due to the importance of this result in economics and optimization theory, several researchers ([8], [9], [11], [13]) obtained the equivalence between the saddle points and the optimal solutions under various conditions on the functions involved.

The purpose of this section is to a define mixed saddle point for a partial Lagrangian of the equivalent optimization problem and to establish the equivalence of the mixed saddle point and optimal solution of (P) under partial $(F,\alpha,\rho,d)-V$ -type I assumption on the function involved.

Definition 6. Vector-valued partial Lagrangian function $L : X \times R_+^{|J_1|} \to R^k$ of the problem (EP) i.e. of (P) is defined as

$$L(x, \mu_{J_1}) = \left\{ L_1(x, \mu_{J_1}), \cdots, L_k(x, \mu_{J_1}) \right\}$$

where $L_i(x, \mu_{J_1}) = f_i(x) + \sum_{j \in J_1} \mu_j g_j(x),\ i = 1, \cdots, k,\ x \in X,\ \mu_{J_1} \in R_+^{|J_1|}$, and $|J_1|$ denotes the cardinality of the index set J_1

Definition 7. A vector $(x^*, \mu_{J_1}^*) \in X \times R_+^{|J_1|}$ is said to be a mixed saddle point for the partial Lagrangian L if

$$L(x^*, \mu_{J_1}^*) \nleqq L(x^*, \mu_{J_1}) \qquad \forall\, \mu_{J_1} \in R_+^{|J_1|}$$

$$L(x, \mu_{J_1}^*) \nleqq L(x^*, \mu_{J_1}^*) \qquad \forall\, x \in X.$$

Theorem 6. Let $(x^*, \lambda^*, \mu^*) \in (S, R_+^k, R_+^m)$ satisfies the necessary conditions (3)-(9). Further let,

(i) $(\lambda_i^* L_i, \mu_j^* g_j)$, $i \in I_1$ and $j \in J_2$ be strictly partially pseudoquasi $(F, \alpha, \rho, d) - V$ -type *I* at x^*.

(ii) $\rho^1 + \rho^2 > 0$,

then $(x^*, \mu_{J_1}^*)$ is a mixed saddle point of *L*.

Where, $I_1 = \{ i \in K : \lambda_i^* > 0 \}$

Proof: Suppose $L_i(x, \mu_{J_1}^*) \leqq L_i(x^*, \mu_{J_1}^*), \qquad i \in I_1$

Since $\lambda_i^* > 0$ and $\alpha_i^1(x, x^*) > 0$, for $i \in I_1$, therefore

$$\sum_{i \in I_1} \alpha_i^1(x, x^*) \lambda_i^* L_i(x, \mu_{J_1}^*) \leqq \sum_{i \in I_1} \alpha_i^1(x, x^*) \lambda_i^* L_i(x^*, \mu_{J_1}^*)$$

Also for $j \in J_2, \alpha_j^2(x, x^*) > 0$, therefore from (5), we have

$$-\sum_{j \in J_2} \alpha_j^2(x, x^*) \mu_j^* g_j(x^*) = 0$$

Using hypothesis (*i*), we have

$$F(x, x^*; \sum_{i \in I_1} \lambda_i^* \nabla L_i(x^*, \mu_{J_1}^*)) < -\rho^1 d^2(x, x^*) \tag{33}$$

$$F(x, x^*; \sum_{j \in J_2} \mu_j^* \nabla g_j(x^*)) \leqq -\rho^2 d^2(x, x^*) \tag{34}$$

We can write (33) as

$$F(x, x^*; \sum_{i \in I_1} \lambda_i^* \nabla f_i(x^*) + \sum_{i \in I_1} \lambda_i^* \sum_{j \in J_1} \mu_j^* \nabla g_j(x^*)) < -\rho^1 d^2(x, x^*)$$

Using $\lambda_i^* = 0,\ i \notin I_1$, above inequality can be rewritten as

$$F(x,x^*;\sum_{i=1}^{k}\lambda_i^*\nabla f_i(x^*)+\sum_{j\in J_1}\mu_j^*\nabla g_j(x^*))<-\rho^1 d^2(x,x^*) \tag{35}$$

Now

$$\begin{aligned}
0 &= F(x,x^*,0)\\
&= F(x,x^*;\sum_{i=1}^{k}\lambda_i^*\nabla f_i(x^*)+\sum_{j=1}^{m}\mu_j^*\nabla g_j(x^*))\\
&= F(x,x^*;\sum_{i=1}^{k}\lambda_i^*\nabla f_i(x^*)+\sum_{j\in J_1}\mu_j^*\nabla g_j(x^*)+\sum_{j\in J_2}\mu_j^*\nabla g_j(x^*))\\
&\le F(x,x^*;\sum_{i=1}^{k}\lambda_i^*\nabla f_i(x^*)+\sum_{j\in J_1}\mu_j^*\nabla g_j(x^*))+F(x,x^*;\sum_{j\in J_2}\mu_j^*\nabla g_j(x^*)) && (\because \text{ F is sublinear})\\
&<-(\rho^1+\rho^2)\,d^2(x,u) && \text{(using (34) and (35))}\\
&<0 && \text{(using assumption (ii))}
\end{aligned}$$

which is not possible. Hence $L(x,\mu_{J_1}^*)\nleqq L(x^*,\mu_{J_1}^*)$.

Also from (5), (7), and (9), we have

$$\begin{aligned}
&\mu_j g_j(x^*)\leqq 0=\mu_j^* g_j(x^*)\\
&\Rightarrow \mu_j g_j(x^*)\leqq\mu_j^* g_j(x^*),\ j\in J_1, i\in K\\
&\Rightarrow f_i(x^*)+\sum_{j\in J_1}\mu_j g_j(x^*)\leqq f_i(x^*)+\sum_{j\in J_1}\mu_j^* g_j(x^*),\ i\in K,\\
&\Rightarrow L(x^*,\mu_{J_1})\leqq L(x^*,\mu_{J_1}^*)\\
&\Rightarrow L(x^*,\mu_{J_1}^*)\nleqq L(x^*,\mu_{J_1}) \qquad \forall\,\mu_{J_1}\in R_+^{|J_1|}
\end{aligned}$$

Theorem 7. If $(x^*,\mu_{\mathrm{j}_1}^*)$ is a mixed saddle point for the partial Lagrangian then x^* is (P)-optimal.

Proof: Since $(x^*,\mu_{\mathrm{j}_1}^*)$ is a mixed saddle point for the partial Lagrangian, then we have

$$\begin{aligned}
&L(x^*,\mu_{\mathrm{j}_1}^*)\nleqq L(x^*,\mu_{J_1}) \qquad \forall\,\mu_{J_1}\in R_+^{|J_1|}\\
&\Rightarrow f_i(x^*)+\sum_{j\in J_1}\mu_j^* g_j(x^*)\nleqq f_i(x^*)+\sum_{j\in J_1}\mu_j g_j(x^*)\\
&\Rightarrow \sum_{j\in J_1}\mu_j^* g_j(x^*)\nleqq\sum_{j\in J_1}\mu_j g_j(x^*)\\
&\Rightarrow (\mu_j^*-\mu_j)\,g_j(x^*)\geqq 0,\ j\in J_1,\ i\in K
\end{aligned} \tag{36}$$

Following on the lines of Mangasrian [8], it follows

$$g_j(x^*)\le 0, \qquad j\in J_1, i\in K$$

$$\therefore\quad \mu_j^* g_j(x^*)\le 0, \qquad j\in J_1, i\in K \tag{37}$$

From (36), if $\mu_j=0,\ j\in J_1$, we get,

$$\mu_j^* g_j(x^*) \geq 0 \tag{38}$$

Therefore, from (37) and (38), we have

$$\begin{aligned} &\mu_j^* g_j(x^*) = 0,\ j \in J_1, i \in K \\ &\Rightarrow \lambda_i^* \mu_j^* g_j(x^*) = 0,\ j \in J_1, i \in K \end{aligned} \tag{39}$$

Since x^* is (P)-feasible therefore there exists $q^* \in R$ such that (x^*, q^*) is (EP)-feasible. Therefore,

$$f_i(x^*) \leq q^*, \qquad \forall\, i \in K$$

$$\Rightarrow \text{either } f_i(x^*) - q^* < 0 \text{ or } f_i(x^*) - q^* = 0$$

Define $I^* = \{i \in I : \lambda_i^* = 0 \text{ when } f_i(x^*) - q^* < 0\}$

$$\therefore \quad \lambda_i^*(f_i(x^*) - q^*) = 0, \qquad \forall\, i \in K \tag{40}$$

From (39) and (40), we have

$$\lambda_i^* \left[f_i(x^*) + \mu_j^* g_j(x^*) \right] = \lambda_i^* q^*, \quad \forall\, i \in K \tag{41}$$

Let us assume on the contrary that x^* is not (P)-optimal then there exist x^0 feasible for (P) and there exists corresponding $q^0 \in R$ such that (x^0, q^0) is (EP)-feasible therefore,

$$f_i(x^0) \leq q^0, \qquad i \in K,$$

and

$$g_j(x^0) \leq 0, \qquad j \in M$$

The above two inequalities imply that

$$\lambda_i^* [f_i(x^0) + \sum_{j \in J_1} \mu_j^* g_j(x^0)] \leqq \lambda_i^* q^0, \qquad i \in K \tag{42}$$

Since, $\quad q^0 < q^* \text{ or } \lambda_i^* q^0 < \lambda_i^* q^*, \qquad i \not\in I^*$

Using (41) and (42) the above inequality become

$$\lambda_i^* [f_i(x^0) + \sum_{j \in J_1} \mu_j^* g_j(x^0)] \leqq \lambda_i^* q^0 < \lambda_i^* q^* = \lambda_i^* [f_i(x^*) + \mu_j^* g_j(x^*)], \qquad i \notin I^*$$

i.e. $\quad \sum_{i=1}^{k} \lambda_i^* L_i(x^0, \mu_{j_1}^*) < \sum_{i=1}^{k} \lambda_i^* L_i(x^*, \mu_{j_1}^*)$

which is a contradiction to the fact that

$$L(x, \mu_{j_1}^*) \nleqq L(x^*, \mu_{j_1}^*) \qquad \forall\, x \in X.$$

Hence, x^* is (P)-optimal.

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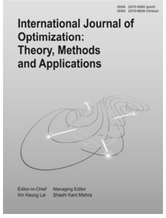

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Improved Particle Swarm Optimization Algorithm for Solving Nonlinear Constrained Optimization Problems ♣

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Abstract. An improved particle swarm optimization algorithm is proposed for solving nonlinear constrained optimization problems. In this algorithm, the speed equation is improved so as to make no memory, and the nonlinear constrained optimization problems is converted unconstrained bi-objection optimization problems by using dynamic bi-objection constraint-handling method. In each iteration, keeping a part of infeasible particles which have good performance aims to maintain the diversity of the population. At last, a mutation operator is employed to expand search field and to overcome premature convergence. It is shown by numerical experiments on the 22 representative benchmarks functions that the proposed algorithm has fast convergence speed and better global optimization ability.

Keywords: global optimization, nonlinear constrained optimization, particle swarm optimization, chaotic, mutation operator.

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1 Introduction

In this paper, we consider the general nonlinear constrained optimization problems (NCOP) below:

$$\begin{cases} \min\ f(x) \\ \text{s. t.}\ \ g_i(x) \le 0 \qquad i = 1, 2, \cdots, q \\ \qquad h_j(x) = 0 \qquad j = q+1, \cdots, m \end{cases} \tag{1}$$

where $x = (x_1, x_2, \cdots x_n) \in R^n$, q is the number of inequality constraints, m is the number of equality constraints. Assume that the search space S be an n−dimensional space bounded by parametric constraints:

$$x_d^l \le x_d \le x_d^u \tag{2}$$

and the feasible region $F \subseteq S$ where

$$F = \{x \mid g_i(x) \le 0, i = 1, 2 \cdots, q; h_j(x) = 0, j = q+1, \cdots m\} \tag{3}$$

The problem (NCOP) comes from many field of science and engineering. Solving nonlinear constrained optimization problems is relatively difficult. In particular, solving its global optimization solution is more difficult. For solving the problems, a class of methods are deterministic ones, for example, the sequential quadratic programming methods [1,2], Lagrange multiplier methods [3] which are usually require the derivative of the objective function and constraints and which can only obtain local optimization solutions, and branch and bound method [4,5], outer approximation method [6,7] which can obtain global optimization solutions. Another types are stochastic methods, for example, genetic algorithm (GA), simulated annealing algorithm (SA), immune algorithm (IA), ant colony algorithms (ACO), particle swarm optimization (PSO), differential evolution algorithm (DE) which are not also require the derivative of the objective function and constraints and which can obtain global optimization solutions.

At present, some constraint-handling techniques are given to be successfully applied to intelligent algorithms for solving nonlinear constrained optimization problem [8]. We know that the penalty function is the most popular constraint handling technique such that original constrained optimization problems are transformed into unconstrained optimization ones. The main difficulty of the penalty function method lies in selecting proper penalty factor which is tightly dependent on the problem itself. Some self-adaptive mechanisms have been introduced into the penalty function method to avoid the trial and error process of tuning factors. Coello [9] presented the notion of co-evolution to adapt the penalty factor of a fitness function in a genetic algorithm. He and Wang[10] proposed a co-evolutionary particle swarm optimization (CPSO),where PSO was used to evolved both feasible solution and penalty factor.

Apart from the penalty function method, several novel techniques have been incorporated into Evolutionary Algorithms (EAs) to handle constraints. Koziel and Michalewicz [11] proposed a homomorphous mapping (HM) between a high dimensional cube and a feasible search space to transform the original problem to be unconstrained. Runarsson. Yao [12] proposed a constraint-handling technique (Stochastic Ranking, SR) from the viewpoint of balancing the dominance between the objective and penalty functions, and the approach focused on the rank of the individuals directly using a bubble-sort like algorithm. He Qie and Ling Wang [13] proposed a hybrid particle swarm optimization with a feasibility-based rule for constrained optimization. Lu and Chen[14] proposed a self-adaptive velocity particle swarm optimization for solving constrained optimization problems using a dynamic-objective constraint-handling method.

In this paper, an improved particle swarm optimization (ICPSO) is proposed to solve nonlinear constrained optimization problems. In this algorithm, the inertia weight is taken as 0 to avoid the difficulties of the experience parameter settings and enhance the local search capabilities. The original problems is converted unconstrained bi-objection optimization problems by using dynamic objective constraint-handling method. In each iteration, keeping a part of the good infeasible particles to maintain the diversity of the population. At last, we introduce a mutation operator in order to expanded the scope of the search space and to overcome premature convergence. Numerical results show that the algorithm is effective and feasible.

The rest of the paper is organized as follows. Section 2 introduces the standard PSO and improvement strategy. Section 3 gives an improved PSO for solving NCOP. Numerical experiments and analysis in Section 4. Finally, the conclusions is given in Section 5.

2 The Standard PSO and Improvement Strategy

2.1 The Standard PSO

Particle swarm optimization (PSO) was proposed by Kennedy and Eberhart [15,16] in 1995 as a simulation of a flock of bird or the sociological behavior of a group of people. In PSO, multiple candidate individuals called particles coexist and collaborate simultaneously and the position of each particle denotes a decision vector for the original problem. The trajectory of each particle in the search space is dynamically adjusted by updating the velocity of each particle, according to its own flying experience and the flying experience of whole particles. Thus PSO combines the local search with the global search so as to balance search mechanism and finally achieves global optimization.

Let n be the dimension of the search space, $x_i = (x_{i1}, x_{i2}, \cdots x_{in})$ be the current position of the i^{th} particle in swarm, $p_i = (p_{i1}, p_{i2}, \cdots, p_{in})$ be the best position of the i^{th} particle so far, and $p_g = (p_{g1}, p_{g2}, \cdots, p_{gn})$ be the best position which of the whole swarm have ever visited. The rate of the velocity for the i^{th} particle is noted as $v_i = (v_{i1}, v_{i2}, \cdots v_{in})$. In original PSO model, the d^{th} dimension position and velocity of the particles are manipulated according to the equations

$$v_{id}(t+1) = wv_{id}(t) + c_1 r_1 (p_{id}(t) - x_{id}(t)) + c_2 r_2 (p_{gd}(t) - x_{id}(t)) \tag{4}$$

$$x_{id}(t+1) = x_{id}(t) + v_{id}(t+1) \tag{5}$$

where $d = 1, 2, \cdots, n$, c_1 and c_2 are positive constants which are called the cognitive and social parameter respectively, both are equal to two generally, r_1 and r_2 are random numbers uniformly distributed in the range [0, 1]. We select $v_{\max}$ and let $|v_{id}| = v_{\max}$ when $|v_{id}| > v_{\max}$, where $v_{\max}$ is a problem-dependent constant. The parameter w is called inertia weight and it can balance global search and local search.

2.2 Improved PSO and Mutation Strategy

In Eqs.(4) and (5), the inertia weight w has the trend of expansion of search space and has not a universal adaptation method. Especially in solving high-dimensional complex constrained optimization problem, if w is too larger, the particles will be make out of feasible region; if w is too smaller, the particles will be fall into local optimal solution. Therefore, the inertia weight w will be set to 0 in this paper, Eqs(4) can be formed as follows:

$$v_{id}(t+1) = c_1 r_1 (p_{id}(t) - x_{id}(t)) + c_2 r_2 (p_{gd}(t) - x_{id}(t)) \tag{6}$$

The particles fly without memory when $w = 0$, the swarm will be shrink to current global optimal position, the global search capabilities will be weaken, while the local search ability will be enhance. The i^{th} particle will stop evolution when $x_i = p_i = p_g$. In order to improve the global search ability of particles, the current particles randomly mutate according to the certain probability p_m below:

$$x_i = x_i(1 + 0.2r) \tag{6}$$

where r is a random number in $(0, 1)$.

2.3 Initialization Chaos Strategy

The PSO's optimization capacity is related with the choice of initial population. If the initial population is uniform distribution in the search space, PSO's global search capability is good.

Because the chaotic variable has ergodicity, pseudo-randomness and irregularity, initial population is produced by chaos strategy. The generated chaos particles are selected by the constraint violation of initialization δ_1 in this paper. The logistic equation is a typical chaotic system[17], which is defined as follows:

$$z_{n+1} = u \cdot z_n (1 - z_n),\ 0 < z_0 < 1 \tag{8}$$

where u is the control parameter, z is a variable and $n = 0, 1, 2, \cdots$. Although the above equation is deterministic, it exhibits chaotic dynamics when $u = 4$ and $z_0 \neq \{0.25, 0.5, 0.75\}$, If $x_i \in (a_i, b_i)$, there is

$$z_i = (x_i - a_i) / (b_i - a_i) \tag{9}$$

$$x_i = a_i + z_i \cdot (b_i - a_i) \tag{10}$$

between the above formula and chaos variable.

The position and velocity of each particle is initialized by using chaos. It can enhance the diversity and ergodicity of particle swarm.

3 An Improved PSO Algorithm for Solving NCOP

3.1 Constraint-handling Technique

Constraint-handling rules are very important in solving constrained optimization problems. we will use dynamic-objective constraint-handling method (DOCHM) [18]: through defining a distance function $\varphi(x)$, DOCHM converts the original problem into a bi-objective optimization problem $\min(\varphi(x), f(x))$ where $\varphi(x)$ is treated as the first objective and $f(x)$ the second one. The distance function $\varphi(x)$ is defined as follows:

$$\varphi(x) = \sum_{i=1}^{q} \max\{0, g_i(x)\} + \sum_{j=q+1}^{m} \max\{0, |g_j(x)|\} \tag{11}$$

Clearly, $\varphi(x)$ is the sum of constraint violations, $\varphi(x) \geq 0$,and thus $\varphi(x) = 0$ for $\forall x \in F$. Therefore, the constrained optimization problems is converted into a unconstrained bi-objective optimization:

$$\min F(x) = \min\ (\varphi(x), f(x)) \tag{12}$$

It can be seen from Eqs.(11) that all the optimal solutions of $\varphi(x)$ constitute the feasible region F of original problem. So if and only $\varphi(x) = 0,$ the particles will take $f(x)$ as its optimization objective. But the optimal solutions can lie in the bound of the feasible region. The infeasible solutions near the optimal solution is helpful for getting the optimal solution. So we set up a threshold $\delta_2 \geq 0$, if a particle satisfy $\varphi(x) \leq \delta_2$, the particle will instead optimize the real objective function $f(x)$. Otherwise, the particle will take $\varphi(x)$ as its optimization objective. The procedure of updating the individual best particle p_i and the global optimal particle p_g is described (Algorithm 1)as follows:

```
    $\varphi_{ibest} = \varphi(p_i), f_{ibest} = f(p_i), \varphi_i = \varphi(x_i)$
    if $\varphi_i < \varphi_{ibest}$ then $p_i \leftarrow x_i, \varphi_{ibest} \leftarrow \varphi_i$ end
    if $\varphi_i < \delta_2$ and $\varphi_{ibest} < \delta_2$ then $f_i = f(x_i)$
        if $f_i < f_{ibest}$ then $p_i \leftarrow x_i, f_{ibest} \leftarrow f_i$ end
end

    $\varphi_{gbest} = \varphi(p_g), f_{gbest} = f(p_g),$
    if $\varphi_i < \varphi_{gbest}$ then $p_g \leftarrow x_i, \varphi_{gbest} \leftarrow \varphi_i$ end
    if $\varphi_g < \delta_2$ and $\varphi_{gbest} < \delta_2$ then
        if $f_i < f_{gbest}$ then $p_g \leftarrow x_i, f_{gbest} \leftarrow f_i$ end
```

```
end
    for $i = 1:N$
        if $rand < p_m$ then $x_i = x_i(1+0.2rand)$ end
end
```

3.2 Handle Technique Violated Dearch Space

Some particles can escape from search space. If a particle escape from the search space $[x_d^l, x_d^u]$, we will assign to the mid-point of the best individual particles p_i and the current global optimal particles p_g. That is, if $x_{id} > x_d^u$ or $x_{id} < x_d^l$, then

$$x_{id} = \frac{1}{2}(p_{id} + p_{gd}) \tag{13}$$

3.3 The Description of Improved Chaotic PSO Algorithm

Now, we describe the proposed algorithm for solving nonlinear constrained optimization problems. This algorithm is denoted as IPSO.

Step 1. (Initialization) set the size of population N, the search space dimension *n*, acceleration factor c_1, c_2, the constraint violation of initialization δ_1, the threshold δ_2, the maximum iteration time *T*, chaotic evolution of iteration M（$M > N$）, set $t=1$.

Step 2. Initialize position and velocity of particles by Chaos .

Step 2-1. Randomly generated *n* dimensional vector z_0 in $(0,1)$, $z_0 \neq \{0.25, 0.5, 0.75\}$ obtain the vectors $z_1, z_2, \cdots z_M$ according to Eqs.(8).

Step 2-2. Converting the chaotic variables z_i to decision variables x_i by Eqs. (10).

Step 2-3. Evaluating the particle constraint violation $\varphi(x_i)$ according to Eqs. (11). If $\varphi(x_i) < \delta_1$, the i^{th} particle joins to the original swarm, otherwise, continue initialization until *N* particles are generated with satisfying $\varphi(x_i) < \delta_1$. The initial velocity of each particle is randomly generated.

Step 3. Set the initial position of each particle p_i if $A = \{x \,|\phi(x) \le \delta_2\} \neq \Phi$ then the global optimal position $p_g = \arg\min_{x_i \in A} f(x_i)$ else $p_g = \arg\min_{x_i \in N} \varphi(x_i)$.

Step 4. Update the velocity and position according to Eqs.(5) and (6), if x_{id} of i^{th} particle escapes from the search space $[x_d^l, x_d^u]$, then x_{id} re-assignment using Eqs.(13).

Step 5. Update the individual optimal position pi and the global optimal position p_g according to algorithm1 (see 3.1).

Step 6. If a stopping criterion is met, then output p_g and its function value; otherwise, let $t = t+1$, go back to step 4.

4 Numerical Experiments and Snalysis

To evaluate the performance of the proposed algorithm, we conducted a series of experiments on the 22 well-known benchmark functions which can seen in the Appendix at the end of this paper. In the test functions, g01 ~ g13 [11,12,14] can be considered "difficult" global optimization problems for an evolutionary algorithm, g14 ~ g22 can be considered the special constrained optimization problems [5,6,7] such as the geometric programming, multiplicative programming, which can be solved by deterministic optimization methods.The main characteristics of g01 ~ g13 benchmark functions are summarized in Table 1.

Table 1. Main characteristics of g01 ~ g13 benchmark functions

Problem	n	Function	ρ	LI	NI	LE	NE
g01	13	quadratic	0.0003	9	0	0	0
g02	20	nonlinear	99.9973	1	1	0	0
g03	10	polynomial	0.0026	0	0	0	1
g04	5	quadratic	27.0079	0	6	0	0
g05	4	cubic	0.0000	2	0	0	3
g06	2	cubic	0.0057	0	2	0	0
g07	10	quadratic	0.0000	3	5	0	0
g08	2	nonlinear	0.8581	0	2	0	0
g09	7	polynomial	0.5199	0	4	0	1
g10	8	linear	0.0020	3	3	0	0
g11	2	quadratic	0.0973	0	0	0	1
g12	3	quadratic	4.7697	0	729	0	0
g13	5	exponential	0.0000	0	0	0	3

In Table 1,each value in the third column indicated by $\rho = \left|F\right| / \left|s\right|$ is an estimate of the ratio between the feasible region and the entire of search space, where $\left|F\right|$ is the number of the feasible solutions and $\left|s\right|$ is the total number of solutions randomly generated. In this work, $\left|s\right| = 1000000$. LI, NI, LE, NE are respectively the number of linear inequality, the number of nonlinear inequality, the number of linear equality and nonlinear equality in the constrained optimization problem.

The main characteristics of g01~g13 benchmark functions are summarized in Table 1. In Table 1, each value in the third column indicated by $\rho = \left|F\right| / \left|s\right|$ is an estimate of the ratio between the feasible region and the entire of search space,where $\left|F\right|$ is the number of the feasible solutions and $|s|$ is the total number of solutions randomly generated. In this work, $|s| = 1000000$.LI,NI,LE,NE representing the number of the linear inequality, nonlinear inequality, linear equality and nonlinear equality in the constrained optimization problem.

The procedure of the proposed algorithm is compiled with Matlab 7.0. In this paper, the experiment parameters set as follows: The swarm size $N = 500$, $c_1 = c_2 = 1.8$, $\delta_1 = 100$ in problem g05, g07 and g13 and $\delta_1 = 2$ in the other problems, the threshold $\delta_2 = 10^{-5}$, the maximum iterations $T = 1000$, chaotic evolution of iteration $M = 1000$. All computations are carried out on personal computer for CPU:2.80GHz, 2.80GHz and RAM:1.00GB at double-precision on FANGZHENG Intel (R). Each experiment runs 20 times independently, the experimental results were obtained by using IPSO algorithm with the above experimental settings, the known optimal (Opt) solution for each problem, the best solution (Best), the median solution (Median), the mean solution (Mean), the worst solution (Worst) and the standard deviation (Std) of the obtained solutions statistics for 20 independently running are summarized in Table 2.

Table 2. Experimental results on 22 benchmark functions by IPSO

Problem	Opt	Best	Median	Mean	Worst	Std
g01	-15	-15.0003	-15.0003	-14.90036	-14.0009	0.2471
g02	0.803619	0.803579	0.754855	0.742780	0.610039	0.0530
g03	1	1.004876	0.995229	0.8922234	0.563142	0.1110
g04	-30665.539	-30665.6196	-30665.61961	-30665.6196	-30665.6196	1.3051e-006
g05	5126.4981	5130.5731	5374.0502	5566.1870	6105.5319	409.1102
g06	-6961.81388	-6961.936827	-6961.936816	-6961.93680	-6961.93668	3.3465e-005
g07	24.306	24.495206	25.15075	25.236760	27.146869	0.6394
g08	0.095825	0.0958250	0.0958250	0.0958250	0.0958250	2.5069e-017
g09	680.630	680.630837	680.64404	680.64824	680.68236	0.0130
g10	7049.3307	7051.9359	7433.4814	7447.8193	7767.8990	149.7157
g11	0.75	0.749900	0.7499	0.75200	0.7809	0.0072
g12	1	1.0000	1.0000	1.0000	1.0000	0
g13	0.0539498	0.23416	0.796679	0.901876	3.10773	0.7269
g14	NA	10122.49313	10122.49327	10122.49328	10122.49340	4.0226e-005
g15	NA	-147.6666	-147.6666	-147.6666	-147.6666	2.9160e-014
g16	NA	460238.29453	460289.2165	460332.13754	460559.22319	88.5815
g17	NA	-0.6600	-0.6600	-0.6600	-0.6600	2.5470e-017
g18	NA	1.1771243396	1.1771243396	1.1771243396	1.1771243396	1.2514e-012
g19	NA	997.66126515	997.661265159	997.661265159	997.661265159	1.1984e-013
g20	NA	275.074283	275.0742838	275.0742838	275.07428384	5.9918e-014
g21	NA	0.210060685	0.210060685	0.210060685	0.210060685	1.3084e-017
g22	NA	60.	60.	60.	60.	0

Problems g02, g03, g08 and g12 are all maximization problems. They were converted into minimization problems by using $-f(x)$. We show from Table 1 that the size of problems g05, g07, and g13 is extremely small (practically zero) and it will difficult to produce the appropriate particles in the chaotic initialization. So we set the constraint violation of initialization $\delta_1 = 100$. We can be seen from Table 2 that the majority optimization problems can be found the optimal solution with the IPSO algorithm and the standard deviation is relatively small. The problem g01 has two local minima -

13.000 and -12.453125, the 20 independent running was trapped into the two local minima. Problems g05, g10 and g13 are among those difficult-to-solve problems for evolutionary algorithm. Specifically, g10 is difficult problem for penalty function method, while g05 and g13 involves equality constraints.

In comparison with the three algorithms: the homomorphous mapping (HM) [11], ASCHEA [19] and Self-adaptive velocity particle swarm optimization (SAVPSO) [14], the results of the comparison are seen in Table 3 ~ 5. In comparison with the deterministic global optimization method in the reference ［5 ~ 7］ is a deterministic global optimization method, the results of the comparison are seen in Table 6 where NA is no published data in the reference.

As is shown in Table 3, IPSO are much better performance than HM in 20 independent runs of the best solution, the mean solution and the worst solution. Problem g05, g13 are complex and difficult to use the evolutionary algorithms, HM algorithm can not calculate result, but IPSO has basically obtained global optimal solution.

Table 4 summarizes the comparison of our algorithm with ASCHEA algorithm. For the problem g02, g03, g10, the computational results are superior to the ASCHEA, the worst solution not provide in the ASCHEA. SAVPSO is a self-adaptive velocity PSO for solving constrained optimization problems, it is proposed in recent years. It can be seen from Table 5, the computational results on the problems g01, g02, g03, g09 and g11 is better than the SAVPSO in terms of best solution, the mean solution and the worst solution. The problems g05, g10, g13 are difficult global optimization problems for an evolutionary algorithms, and the size of feasible region is extremely small. So we set the constraint violation of initialization $\delta_1 = 100$ in the chaotic initialization. The error of the IPSO is larger, but the computational results of IPSO are better than the HM, ASCHEA.

Table 3. Comparison between IPSO and HM [11]

Problem	Opt	Best		Mean		Worst	
		IPSO	HM	IPSO	HM	IPSO	HM
g01	-15	-15.0003	-14.7886	-14.990036	-14.7082	-14.0009	-14.61540
g02	0.803619	0.803579	0.799530	0.742780	0.796710	0.610039	0.791190
g03	1	1.004876	0.999700	0.8922234	0.998900	0.563142	0.997800
g04	-30665.539	-30665.61	-30664.500	-30665.619	-30655.300	-30665.619	-30645.9000
g05	5126.4981	5130.5731	NA	5566.1870	NA	6105.5319	NA
g06	-6961.81388	-6961.936	-6952.100	-6961.9368	-6342.6000	-6961.9366	-5473.9000
g07	24.306	24.495206	24.62000	25.236760	24.826000	27.146869	25.069000
g08	0.095825	0.0958250	0.095825	0.0958250	0.089157	0.0958250	0.029144
g09	680.630	680.63083	680.91000	680.64824	681.16000	680.64824	683.180000
g10	7049.3307	7051.9359	7147.90000	7447.8193	8163.60000	7767.8990	9659.300000
g11	0.75	0.749900	0.75000	0.75200	0.75000	0.7809	0.75.0000
g12	1	1.0000	1.0000	1.0000	0.999135	1.0000	0.991950
g13	0.0539498	0.23416	NA	0.901876	NA	3.10773	NA

Table 4. Comparison between IPSO and ASCHEA [19]

Problem	Opt	Best		Mean		Worst	
		IPSO	ASCHEA	IPSO	ASCHEA	IPSO	ASCHEA
g01	-15	-15.0003	-15.00000	-14.90036	-15.0000	-14.0009	NA
g02	0.803619	0.803579	0.785000	0.742780	0.5900000	0.610039	NA
g03	1	1.004876	1.00000	0.8922234	0.999890	0.563142	NA
g04	-30665.539	-30665.6196	30665.5000	-30665.6196	-30665.500	-30665.619	NA
g05	5126.4981	5130.5731	5126.5000	5566.1870	5141.65000	6105.5319	NA
g06	-6961.81388	-6961.936	-6961.8100	-6961.9368	-6961.8100	-6961.9366	NA
g07	24.306	24.495206	24.332300	25.236760	24.660000	27.146869	NA
g08	0.095825	0.0958250	0.098825	0.0958250	0.095825	0.0958250	NA
g09	680.630	680.63083	680.6300	680.64824	680.64100	680.64824	NA
g10	7049.3307	7051.9359	7061.13000	7447.8193	7193.1100	7767.8990	NA
g11	0.75	0.749900	0.75000	0.75200	0.75000	0.7809	NA
g12	1	1.0000	NA	1.0000	NA	1.0000	NA
g13	0.0539498	0.23416	NA	1.0000	NA	3.10773	NA

Table 5. Comparison between IPSO and SAVPSO[14]

Problem	Opt	Best		Mean		Worst	
		IPSO	SAVPSO	IPSO	SAVPSO	IPSO	SAVPSO
g01	-15	-15.0003	-15	-14.90036	-14.7151	-14.0009	-12.45
g02	0.803619	0.803579	0.803443	0.742780	0.740577	0.610039	0.631
g03	1	1.004876	1.0048	0.8922234	1.0034	0.563142	0.9976
g04	- 30665.539	-30665.6196	-30665.539	-30665.6196	-30665.53	-30665.619	-30665.539
g05	5126.4981	5130.5731	5126.4842	5566.1870	5202.3627	6105.5319	5520.1467
g06	-6961.81388	-6961.93682	-6961.81388	-6961.9368	-6961.813	-6961.9366	-6961.8138
g07	24.306	24.495206	24.319	25.236760	24.989	27.146869	26.194
g08	0.095825	0.0958250	0.095825	0.0958250	0.095825	0.0958250	0.95825
g09	680.630	680.63083	680.632	680.64824	680.653	680.64824	680.699
g10	7049.3307	7051.9359	7054.1256	7447.8193	7173.2661	7767.8990	7335.24
g11	0.75	0.749900	0.749	0.75200	0.749	0.7809	0.749
g12	1	1.0000	1	1.0000	1.0	1.0000	1.0
g13	0.0539498	0.23416	0.053866	1.0000	0.552753	3.10773	1.85610

Table 6. Comparison between IPSO and references [5-7]

Problem	Reference[5]	Reference [6]	Reference [7]	IPSO
g14	10122.38112168(10^{-9})	NA	10122.493176362(0)	10122.49313(0)
g15	-83.249728406(0)	NA	-83.249728406(0)	-147.66666(0)
g16	623249.87529475(10^{-9})	NA	623249.87611810(0)	460238.294533(0)
g17	-1.99(0)	NA	NA	-0.6600(0)
g18	1.177124327(0)	NA	NA	1.1771243396(0)
g19	NA	997.661265(0)	NA	997.661265159(0)
g20	NA	275.074284(0)	NA	275.074283849(0)
g21	NA	0.227609428(0)	NA	0.21006068580(0)
g22	NA	60.0(0)	NA	60.0(0)

The benchmark functions g14 ~ g22 are from the reference ［5 ~ 7］ where the deterministic global optimization solutions are obtained. In comparison with the deterministic global optimization ［5 ~ 7］, the results is seen in Table 6, where (∗) is the optimal solution of the constraint violation. It can be seen from the Table 6 that the performance of IPSO algorithm is basically better than the deterministic global optimization.

In particular, the problems g15, g16, g21 have been new optimal value with IPSO, and their optimal solution (150.0000, 30.0000, 4.1343), (43.1494, 44.9833, 70.0000, 1.2155), (1.8000, 2.0000), respectively. It has better the reported experimental results.

5. Conclusion

This paper presents an improved particle swarm optimization for solving nonlinear constrained optimization problems. In the new algorithm, the inertia weight is set as 0, so as to enhance local search ability near the optimal points. In order to get the optimal solution near the constrained bounder, we define a constraint violation function and retains a part of the performance of better infeasible particles according to a certain threshold. The given mutation operator is introduced to enhance global search capability and is avoid premature stagnation of the particles. The nonlinear constrained optimization problems is converted unconstrained bi-objection optimization problems by using dynamic bi-objection constraint-handling method. Numerical experiments show that the proposed algorithm is effectiveness, versatility, robustness and global optimization.

Appendix

All benchmark functions used in the paper are summarized below:

1. g01

$$\begin{cases} \min\ f(x) = 5\sum_{i=1}^{4} x_i - 5\sum_{i=1}^{4} x_i^2 - \sum_{i=5}^{13} x_i \\ \text{s.t.}\quad g_1(x) = 2x_1 + 2x_2 + x_{10} + x_{11} - 10 \le 0 \\ \qquad g_2(x) = 2x_1 + 2x_3 + x_{10} + x_{12} - 10 \le 0 \\ \qquad g_3(x) = 2x_2 + 2x_3 + x_{11} + x_{12} - 10 \le 0 \\ \qquad g_4(x) = -8x_1 + x_{10} \le 0, \\ \qquad g_5(x) = -8x_2 + x_{11} \le 0 \\ \qquad g_6(x) = -8x_3 + x_{12} \le 0, \\ \qquad g_7(x) = -2x_4 - x_5 + x_{10} \le 0 \\ \qquad g_8(x) = -2x_6 - x_7 + x_{11} \le 0, \\ \qquad g_9(x) = -2x_8 - x_9 + x_{12} \le 0 \end{cases}$$

where the bounds are $0 \le x_i \le 1 (i = 1, 2, \cdots 9)$, $0 \le x_i \le 100 (i = 10, 11, 12)$ and $0 \le x_{13} \le 10$. The global optimum is at $x^* = (1,1,1,1,1,1,1,1,1,3,3,3,1)$ where $f(x^*) = -15$.

2. g02

$$\begin{cases} \max\ f(x) = \left| \dfrac{\sum_{i=1}^{n} \cos^4(x_i) - 2\prod_{i=1}^{n} \cos^2(x_i)}{\sqrt{\sum_{i=1}^{n} i x_i^2}} \right| \\ \text{s.t.}\quad g_1(x) = 0.75 - \prod_{i=1}^{n} x_i \le 0 \\ \qquad g_2(x) = \sum_{i=1}^{n} x_i - 7.5n \le 0 \end{cases}$$

where $n = 20$ and $0 \le x_i \le 10$ $(i = 1, 2, \cdots n)$. The global maximum is unknown; the best reported solution is $f(x^*) = 0.83619$.

3. g03

$$\begin{cases} \max\ f(x) = (\sqrt{n})^n \prod_{i=1}^{n} x_i \\ \text{s.t.}\quad h(x) = \sum_{i=1}^{n} x_i^2 - 1 = 0 \end{cases}$$

where $n = 10$ and $0 \le x_i \le 1 (i = 1, 2, \cdots n)$. The global maximum is at $x^* = 1/\sqrt{n} (i = 1, \cdots n)$ where $f(x^*) = 1$.

4. g04

$$\begin{cases} \min\ f(x) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141 \\ \text{s.t.}\ \ g_1(x) = 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 - 92 \le 0, \\ \quad g_2(x) = -85.334407 - 0.0056858x_2x_5 - 0.0006262x_1x_4 + 0.0022053x_3x_5 \le 0, \\ \quad g_3(x) = 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3^2 - 110 \le 0, \\ \quad g_4(x) = -80.51249 - 0.0071317x_2x_5 - 0.0029955x_1x_2 - 0.0021813x_3^2 + 90 \le 0, \\ \quad g_5(x) = 9.300961 + 0.0047062x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 - 25 \le 0, \\ \quad g_6(x) = -9.300961 - 0.0047062x_3x_5 - 0.0012547x_1x_3 - 0.0019085x_3x_4 + 20 \le 0,, \end{cases}$$

where $78 \le x_1 \le 102$, $33 \le x_2 \le 45$, $27 \le x_i \le 45$ $(i = 3, 4, 5)$. The global optimum is at $x^* = (78, 33, 29.995256025682, 45, 36.775812905788)$ where $f(x^*) = -30665.539$.

5. g05

$$\begin{cases} \min\ f(x) = 3x_1 + 0.000001x_1^3 + 2x_2 + (0.000002/3)x_2^3 \\ \text{s.t.}\ \ g_1(x) = x_4 - x_3 + 0.55 \ge 0 \\ \quad g_2(x) = -x_4 + x_3 + 0.55 \ge 0 \\ \quad h_3(x) = 1000\sin(-x_3 - 0.25) + 1000\sin(-x_4 - 0.25) + 894.8 - x_1 = 0 \\ \quad h_4(x) = 1000\sin(x_3 - 0.25) + 1000\sin(x_3 - x_4 - 0.25) + 894.8 - x_2 = 0 \\ \quad h_5(x) = 1000\sin(x_4 - 0.25) + 1000\sin(x_4 - x_3 - 0.25) + 1294.8 = 0 \end{cases}$$

where $0 \le x_i \le 1200$ $(i = 1.2)$, $-0.55 \le x_i \le 0.55$ $(i = 3, 4)$. The best known solution is $x^* = (679.9543, 1026.067, 0.1188764, -0.3962336)$, where $f(x^*) = 5126.4981$.

6. g06

$$\begin{cases} \min\ f(x) = (x_1 - 10)^3 + (x_2 - 20)^3 \\ \text{s.t.}\ \ g_1(x) = -(x_1 - 5)^2 - (x_2 - 5)^2 + 100 \le 0 \\ \quad g_2(x) = (x_1 - 6)^2 + (x_2 - 5)^2 - 82.81 \le 0 \end{cases}$$

where $13 \le x_1 \le 100$; $0 \le x_2 \le 100$. The optimum solution is $x^* = (14.095, 0.84296)$ where $f(x^*) = -6961.81388$.

7. g07

$$\begin{cases} \min\ f(x) = x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 \\ \qquad + 2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45 \\ \text{s.t.}\ \ g_1(x) = 105 - 4x_1 - 5x_2 + 3x_7 - 9x_8 \ge 0 \\ \quad g_2(x) = -3(x_1 - 2)^2 - 4(x_2 - 3)^2 - 2x_3^2 + 7x_4 + 120 \ge 0 \\ \quad g_3(x) = -10x_1 + 8x_2 + 17x_7 - 2x_8 \ge 0 \\ \quad g_4(x) = -x_1^2 - 2(x_2 - 2)^2 + 2x_1x_2 - 14x_5 + 6x_6 \ge 0 \\ \quad g_5(x) = 8x_1 - 2x_2 - 5x_9 + 2x_{10} + 12 \ge 0 \\ \quad g_6(x) = -5x_1^2 - 8x_2 - (x_3 - 6)^2 + 2x_4 + 40 \ge 0 \\ \quad g_7(x) = 3x_1 - 6x_2 - 12(x_9 - 8)^2 + 7x_{10} \ge 0 \\ \quad g_8(x) = -0.5(x_1 - 8)^2 - 2(x_2 - 4)^2 - 3x_5^2 + x_6 + 30 \ge 0 \end{cases}$$

where $-10.0 \le x_i \le 10.0\,(i = 1, 2, \cdots 10)$. The global optimum is $x^* = (2.171996, 2.363683, 8.773926, 5.095984, 0.9906548, 1.430574, 1.321644, 9.828726, 8.280092, 8.375927)$ where $f(x^*) = 24.3062091$.

8. g08

$$\begin{cases} \max\ f(x) = \dfrac{\sin^3(2\pi x_1)\sin(2\pi x_2)}{x_1^{\,3}(x_1 + x_2)} \\ \text{s.t.}\quad g_1(x) = x_1^{\,2} - x_2 + 1 \le 0 \\ \qquad\ \ g_2(x) = 1 - x_1 + (x_2 - 4)^2 \le 0 \end{cases}$$

where $0 \le x_1 \le 10$; $0 \le x_2 \le 10$. The optimum solution is located at $x^* = (1.227913, 4.2453733)$ where $f(x^*) = 0.095825$.

9. g09

$$\begin{cases} \min\ f(x) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^{\,4} + 3(x_4 - 11)^2 + 10x_5^{\,6} + 7x_6^{\,2} + x_7^{\,4} - 4x_6x_7 - 10x_6 - 8x_7 \\ \text{s.t.}\quad g_1(x) = 127 - 2x_1^{\,2} - 3x_2^{\,4} - x_3 - 4x_4^{\,2} - 5x_5 \ge 0, \\ \qquad\ \ g_2(x) = 282 - 7x_1 - 3x_2 - 10x_3^{\,2} - x_4 + x_5 \ge 0, \\ \qquad\ \ g_3(x) = 196 - 23x_1 - x_2^{\,2} - 6x_6^{\,2} + 8x_7 \ge 0, \\ \qquad\ \ g_4(x) = -4x_1^{\,2} - x_2^{\,2} + 3x_1x_2 - 2x_3^{\,2} - 5x_6 + 11x_7 \ge 0 \end{cases}$$

where $-10.0 \le x_i \le 10.0\,(i = 1, 2, \cdots 7)$. The optimum solution is $x^* = (2.330499, 1.951372, -0.4775414, 4.365726, -0.6244870, 1.038131, 1.594227)$ where $f(x^\bullet) = 680.6300573$.

10. g10

$$\begin{cases} \min\ f(x) = x_1 + x_2 + x_3 \\ \text{s.t.}\quad g_1(x) = -1 + 0.0025(x_4 + x_6) \le 0 \\ \qquad\ \ g_2(x) = -1 + 0.0025(x_5 + x_7 - x_4) \le 0 \\ \qquad\ \ g_3(x) = -1 + 0.01(x_8 - x_5) \le 0 \\ \qquad\ \ g_4(x) = -x_1x_6 + 833.33252x_4 + 100x_1 - 83333.333 \le 0 \\ \qquad\ \ g_5(x) = -x_2x_7 + 1250x_5 + x_2x_4 - 1250x_4 \le 0 \\ \qquad\ \ g_6(x) = -x_3x_8 + 1250000 + x_3x_5 - 2500x_5 \le 0 \end{cases}$$

where $100 \le x_1 \le 10000, 1000 \le x_i \le 10000\,(i = 2, 3)$ and $10 \le x_i \le 1000\,(i = 4, \cdots 8)$. The global optimum is $x^* = (579.3167, 1359.943, 5110.071, 182.0174, 295.5985, 217.9799, 286.4162, 395.5979)$ where $f(x^*) = 7049.3307$

11. g11

$$\begin{cases} \min\ f(x) = x_1^{\,2} + (x_2 - 1)^2 \\ \text{s.t.}\quad h_1(x) = x_2 - x_1^{\,2} = 0 \end{cases}$$

where $-1 \le x_1 \le 1$; $-1 \le x_2 \le 1$. The optimum solution is $x^* = (\pm 1/\sqrt{2}, 1/2)$ where $f(x^*) = 0.75$.

12. g12

$$\begin{cases} \max\ f(x) = (100 - (x_1 - 5)^2 - (x_2 - 5)^2 - (x_3 - 5)^2)/100 \\ \text{s.t.}\quad g(x) = (x_1 - p)^2 + (x_2 - q)^2 + (x_3 - r)^2 - 0.0625 \le 0 \end{cases}$$

where $1 \le x_i \le 10\,(i = 1, 2, 3)$ and $p, q, r = 1, 2, \cdots, 9$. The feasible region of the search space consists of 9^3 disjointed spheres. A point (x_1, x_2, x_3) is feasible if and only if there exist p, q, r such that the above inequality holds. The optimum solution is $x^* = (5, 5, 5)$ where $f(x^*) = 1$.

13. g13

$$\begin{cases} \min\ f(x) = e^{x_1 x_2 x_3 x_4 x_5} \\ \text{s.t.}\quad h_1(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 10 = 0 \\ \qquad\ \ h_2(x) = x_2 x_3 - 5 x_4 x_5 = 0, \\ \qquad\ \ h_3(x) = x_1^3 + x_2^3 + 1 = 0 \end{cases}$$

where $-2.3 \le x_i \le 2.3\,(i = 1, 2)$ and $-3.2 \le x_i \le 3.2\,(i = 3, 4, 5)$. The optimum solution is $x^* = (-1.717143, 1.595709, 1.827247, -0.7636413, -0.763645)$ where $f(x^\bullet) = 0.0539498$.

14. g14[5、7]

$$\begin{cases} \min\ \ f(x) = 5.3578 x_3^2 + 0.8357 x_1 x_5 + 37.2392 x_1 \\ \text{s.t.}\quad g_1(x) = 0.00002584 x_3 x_5 - 0.00006663 x_2 x_5 - 0.0000734 x_1 x_4 \le 1 \\ \qquad\ \ g_2(x) = 0.000853007 x_2 x_5 - 0.00009395 x_1 x_4 - 0.00033085 x_3 x_5 \le 1 \\ \qquad\ \ g_3(x) = 1330.3294 x_2^{-1} x_5^{-1} - 0.42 x_1 x_5^{-1} - 0.30586 x_2^{-1} x_3^2 x_5^{-1} \le 1 \\ \qquad\ \ g_4(x) = 0.00024186 x_2 x_5 + 0.00010159 x_1 x_2 + 0.00007379 x_3^2 \le 1 \\ \qquad\ \ g_5(x) = 2275.1327 x_3^{-1} x_5^{-1} - 0.2668 x_1 x_5^{-1} - 0.40584 x_4 x_5^{-1} \le 1 \\ \qquad\ \ g_6(x) = 0.00029955 x_3 x_5 + 0.00007992 x_1 x_3 + 0.00012157 x_3 x_4 \le 1 \end{cases}$$

where $78.0 \le x_1 \le 102.0, 33.0 \le x_2 \le 45.0, 27.0 \le x_i \le 45.0 (i = 3, 4, 5)$.

15. g15[5、7]

$$\begin{cases} \min\ \ f(x) = 0.5 x_1 x_2^{-1} - x_1 - 5 x_2^{-1} \\ \text{s. t.}\ \ g_1(x) = 0.01 x_2 x_3^{-1} + 0.01 x_2 + 0.0005 x_1 x_3 \le 1 \end{cases}$$

where $70 \le x_1 \le 150, 1 \le x_2 \le 30, 0.5 \le x_3 \le 21$.

16. g16[5、7]

$$\begin{cases} \min\ \ f(x) = 168 x_1 x_2 + 3651.2 x_1 x_2 x_3^{-1} + 40000 x_4^{-1} \\ \text{s.t.}\quad g_1(x) = 1.0425 x_1 x_2^{-1} \le 1 \\ \qquad\ \ g_2(x) = 0.00035 x_1 x_2 \le 1 \\ \qquad\ \ g_3(x) = 1.25 x_1^{-1} x_4 + 41.63 x_1^{-1} \le 1 \end{cases}$$

where $40 \le x_1 \le 44, 40 \le x_2 \le 45, 60 \le x_3 \le 70, 0.1 \le x_4 \le 1.4$.

17. g17[5]

$$\begin{cases} \min \ f(x) = -2x_1^{-1} + x_2^{-1} \\ \text{s.t.} \quad g_1(x) = x_1 + x_2^{-1} \le 10 \\ \qquad\ \ g_2(x) = -x_1 + 2x_2^{-1} \le 8 \\ \qquad\ \ g_3(x) = -2x_1 - 3x_2^{-1} \le -6 \\ \qquad\ \ g_4(x) = x_1 - x_2 \le 4 \\ \qquad\ \ g_5(x) = -2x_1^2 + x_1 x_2 - x_2^2 + 6x_1 \le 1 \end{cases}$$

where $1.0 \le x_1 \le 100.0, 1.0 \le x_2 \le 100.0$.

18. g18[5]

$$\begin{cases} \min \ f(x) = x_1 \\ \text{s.t.} \quad g_1(x) = -\frac{1}{4}x_1 + \frac{1}{2}x_2 - \frac{1}{16}8x_1^2 \le 1 \\ \qquad\ \ g_2(x) = \frac{1}{14}x_1^2 + \frac{1}{14}x_2^2 - \frac{3}{7}x_1 - \frac{3}{7}x_2 \le -1 \end{cases}$$

where $1 \le x_1 \le 5.5, 1 \le x_2 \le 5.5$.

19. g19[6]

$$\begin{cases} \min f(x) = (x_1 + x_2 + 1)^{2.5}(2x_1 + x_2 + 1)^{1.1}(x_1 + 2x_2 + 1)^{1.9} \\ \text{s.t.} \quad g_1(x) = (x_1 + 2x_2 + 1)^{1.1}(2x_1 + 2x_2 + 2)^{1.3} \le 50 \end{cases}$$

where $1 \le x_1 \le 3, 1 \le x_2 \le 3$.

20. g20[6]

$$\begin{cases} \min \ f(x) = (2x_1 + x_2 + 1)^{1.5}(2x_1 + x_2 + 1)^{2.1}(0.5x_1 + 2x_2 + 1)^{0.5} \\ \text{s.t.} \quad g_1(x) = (x_1 + 2x_2 + 1)^{1.2}(2x_1 + 2x_2 + 2)^{0.1} \le 18 \\ \qquad\ \ g_2(x) = (1.5x_1 + 2x_2 + 1)(2x_1 + 2x_2 + 1)^{0.5} \le 25 \end{cases}$$

where $1 \le x_1 \le 3, 1 \le x_2 \le 3$.

21. g21[6]

$$\begin{cases} \min f(x) = 169 \times (x_1 + x_2 + 1) / (37x_1 + 73x_2 + 13) \times (x_1 + 2x_2 + 1) / (63x_1 - 18x_2 + 39) \\ \text{s.t.} \quad g_1(x) = 5x_1 - 3x_2 \le 3 \end{cases}$$

where $1.5 \le x_1 \le 3, 2 \le x_2 \le 3.5$.

22. g22[6]

$$\begin{cases} \min f(x) = (x_1 + x_2 + x_3)(2x_1 + x_2 + x_3)(x_1 + 2x_2 + 2x_3) \\ \text{s.t.} \quad g_1(x) = (x_1 + 2x_2 + x_3)^{1.1}(2x_1 + 2x_2 + x_3)^{1.3} \le 100 \end{cases}$$

where $1 \le x_1 \le 3, 1 \le x_2 \le 3, 1 \le x_2 \le 3$.

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OBJECTIVES

Optimization is a multi-disciplinary research field that deals with the characterization and computation of minima and / or maxima (local/global) of nonlinear, nonconvex, nonsmooth and discrete functions. optimization problems are frequently encountered in modeling of complex real world systems for a very broad range of applications including industrial and systems engineering, management science, operational research, mathematical economics, seismic optimization, production planning and scheduling, transportation and logistics and many other applied areas of science and engineering.

Due to rapid development and deployment of practical optimization methodologies in the last three decades, scientists around the globe in diverse disciplines have been using optimization techniques and algorithms to solve their respective problems. Optimization is playing a pivoting role in the development of modern science and engineering. The main aim of this conference is to provide impetus and motivation for further research work in all areas of optimization and its applications by providing a forum for the academic exchange of ideas and recent research works, both by senior and young active researchers from around the globe.

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All topics related to optimization theory, methods and applications, including, but not limited to:

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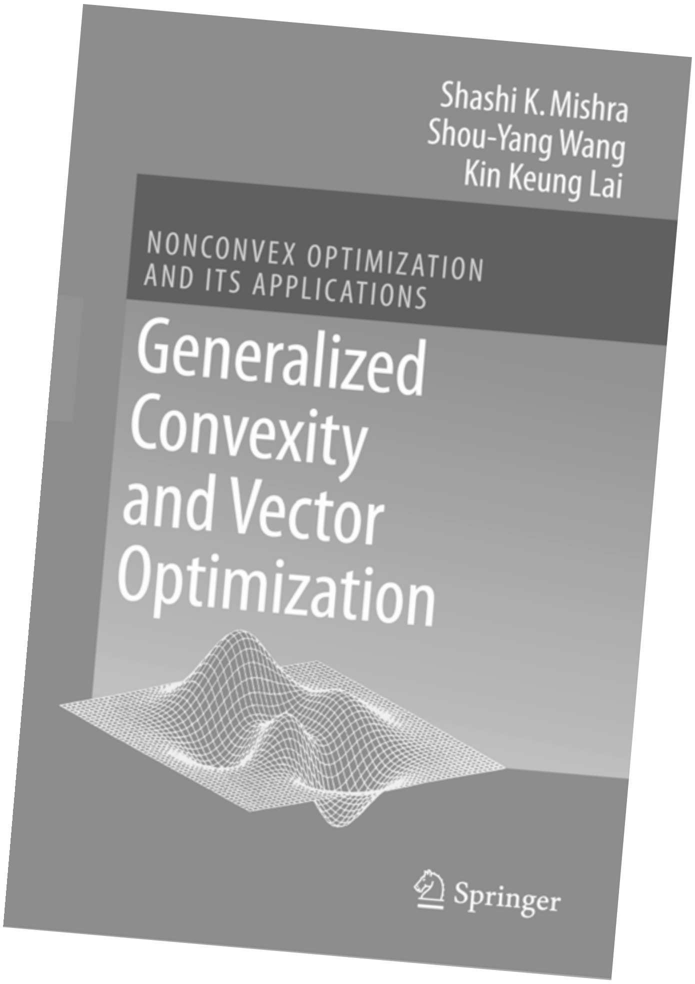

About this book

The present book discusses the Kuhn-Tucker Optimality, Karush-Kuhn-Tucker Necessary and Sufficient Optimality Conditions in presence of various types of generalized convexity assumptions. Wolfe-type Duality, Mond-Weir type Duality, Mixed type Duality for Multiobjective optimization problems such as Nonlinear programming problems, Fractional programming problems, Nonsmooth programming problems, Nondifferentiable programming problems, Variational and Control problems under various types of generalized convexity assumptions.

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V-INVEX FUNCTIONS AND VECTOR OPTIMIZATION summarizes and synthesizes an aspect of research work that has been done in the area of Generalized Convexity over the past several decades. Specifically, the book focuses on V-invex functions in vector optimization that have grown out of the work of Jeyakumar and Mond in the 1990's. V-invex functions are areas in which there has been much interest because it allows researchers and practitioners to address and provide better solutions to problems that are nonlinear, multi-objective, fractional, and continuous in nature. Hence, V-invex functions have permitted work on a whole new class of vector optimization applications.

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