η-Pseudolinearity and Efficiency

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Abstract: Given a nonlinear vector-valued programming problem involving generalized pseudolinear functions (i.e. η-pseudolinear functions) it is shown that every efficient solution is properly efficient under some boundedness condition.

Key words: nonlinear programming, η-pseudoconvex function, η-pseudolinear function, efficiency, proper efficiency.

1 Introduction

A real-valued differentiable function defined on an open set $D$ in $\mathbb{R}^n$ is said to be pseudolinear if $f$ and $-f$ are pseudoconvex. Hanson [5] introduced the class of functions $f$ with the following property:

$$f(y) - f(x) \geq \nabla f(x)^\top \eta(y, x) \text{ for all } x, y \in D,$$

for a vector function $\eta(y, x)$, as a generalization of convex functions. Later those functions were known as η-convex or invex. Hanson [5] also introduced a more general class of functions defined as follows:

$$\nabla f(x)^\top \eta(y, x) \geq 0 \text{ implies } f(y) \geq f(x) \text{ for all } x, y \in D.$$

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Such functions were later called $\eta$-pseudoconvex or pseudoinvex. The reader may consult Mishra and Giorgi [7] for a recent survey of invex functions and their generalizations.

If $\eta(y,x) = y - x$, then the definitions of $\eta$-convexity and $\eta$-pseudoconvexity reduce to the definitions of convexity and pseudoconvexity, respectively. We note that, unlike convex and pseudoconvex functions, the class of invex functions and the class of pseudoinvex functions coincide.

**Definition 1.** (Ansari et al. [1]). A differentiable function $f$ defined on an open set $D$ in $\mathbb{R}^n$ is called $\eta$-pseudolinear if $f$ and $-f$ are $\eta$-pseudoconvex with respect to the same $\eta$.

Every pseudolinear function is $\eta$-pseudolinear with respect to $\eta(y,x) = y - x$, but the converse is not true. Ansari et al. [1] gave an example of functions $f$ and $\eta$ that showed that a function $f$ can be $\eta$-pseudolinear without being pseudolinear.

Chew and Choo [2] obtained first and second order characterizations of pseudolinear functions and found conditions for an efficient solution of a nonlinear vector-valued programming problem to be proper efficient. Kaul et al. [4] extended the class of pseudolinear functions to semilocally pseudolinear functions and discussed conditions of efficiency and properly efficiency for a multiobjective programming problem. Later Ansari et al. [1] obtained the following first order characterizations of $\eta$-pseudolinear functions, generalizing some of the results obtained by Chew and Choo [2]. First we need the following definition, due to Mohan and Neogy [8]. The vector-valued function $\eta : D \times D \to \mathbb{R}^n$, satisfies condition C if for any $x, y \in D$,

\[
\eta(x, y + \lambda \eta(x, y)) = -\lambda \eta(y, x),
\]

\[
\eta(y, x + \lambda \eta(y, x)) = (1 - \lambda) \eta(y, x)
\]

for all $\lambda \in [0,1]$.

Suppose that $f : D \to \mathbb{R}$ is $\eta$-pseudolinear, with $\eta$ satisfying condition C. Then for all $x, y \in D$, $\nabla f(x)\T \eta(y, x) = 0$ if and only if $f(y) = f(x)$.

A differentiable function $f : D \to \mathbb{R}$ is $\eta$-pseudolinear if and only if there exists a function $p$, called proportional functional, defined on $D \times D$ such that $p(x, y) > 0$ and $f(y) = f(x) + p(x, y)$ $\nabla f(x)\T \eta(y, x)$ for all $x, y \in D$.

In this paper we are going to consider the following multiobjective programming problem:

maximize $f(x) = (f_i(x), \ldots, f_j(x))$, subject to $g_j(x) \leq h_j$, $j = 1, \ldots, m$, involving $\eta$-pseudolinear functions $f_i$ and $g_j$, and we are going to show that for a feasible point $x^0$ in the feasible region to be efficient it is necessary and sufficient that the Kuhn-Tucker conditions hold for the function $\lambda_1 f_1 + \cdots + \lambda_k f_k$ for some positive multipliers $\lambda_1, \ldots, \lambda_k$. That is, there exist multipliers $\mu_1, \ldots, \mu_m \geq 0$ such that

\[
\lambda_1 \nabla f_1(x^0) + \cdots + \lambda_k \nabla f_k(x^0) = \mu_1 \nabla g_1(x^0) + \cdots + \mu_m \nabla g_m(x^0),
\]

and

\[
\mu_j (g_j(x^0) - h_j) = 0, j = 1, \ldots, m.
\]

We are also going to show that every efficient solution that satisfies a certain boundedness condition is properly efficient.
2 Efficiency

Consider the following multiobjective $\eta$-pseudolinear programming problem:

\[
(P) \quad \text{V-maximize } f(x) = \left( f_1(x), \ldots, f_k(x) \right) \\
\text{subject to } g_j(x) \leq b_j, \ j = 1, \ldots, m,
\]

where the differentiable functions $f_i$ and $g_j$ are $\eta$-pseudolinear on the open set $D \subseteq \mathbb{R}^n$, with proportional functionals $p_j$ and $q_j$, respectively. Let $X$ be the set of feasible points for problem (P).

**Definition 2.** A feasible point $x$ is said to be an efficient solution of (P) if $f_i(x) \leq f_i(y)$ for all feasible $y$. In other words, there is no other feasible $y$ such that, for some $i = 1, 2, \ldots, k$, we have $f_i(x) < f_i(y), f_j(x) \leq f_j(y), \forall s \neq i$.

Geoffrion [3] introduced the concept of proper efficiency for the maximization problem V-maximize $f(x)$ subject to $x \in X \subseteq \mathbb{R}^n$.

**Definition 3.** A feasible point $x$ is properly efficient if it is efficient and there exists a real number $M > 0$ such that, for each $i$, we have

\[
f_i(y) - f_i(x) \leq M (f_j(x) - f_j(y))
\]

for some $j$ such that $f_j(x) > f_j(y)$ whenever $f_j(y) > f_i(x)$.

The following result generalizes Proposition 3.2 of (Chew and Choo [2]) to $\eta$-pseudolinearity.

**Proposition 1.** Consider problem (P) where the differentiable functions $f_i$ and $g_j$ $(i = 1, \ldots, k; j = 1, \ldots, m)$ are $\eta$-pseudolinear on the set $D \subseteq \mathbb{R}^n$ with proportional functionals $p_j$ and $q_j$, respectively. Let condition C be satisfied for all $x, y \in D$. A feasible point $x^0$ is an efficient solution of (P) if and only if there exist multipliers $\lambda_i > 0$ and $\mu_j \geq 0$, $i = 1, 2, \ldots, k, j \in I(x^0) = \{ j \mid g_j(x^0) = b_j \}$ such that

\[
\sum_{i=1}^{k} \lambda_i \nabla f_i(x^0) = \sum_{j \in I(x^0)} \mu_j \nabla g_j(x^0).
\]

**Proof:** Suppose that there exist $\lambda_i$ and $\mu_j$ that satisfy (1), but $x^0$ is not efficient. Then there exists a feasible point $y$ such that $f_i(x^0) \leq f_i(y)$ for all $i$ and $f_s(x^0) < f_s(y)$ for some $s$. Then

\[
0 \geq \sum_{j \in I(x^0)} \frac{\mu_j (g_j(y) - g_j(x^0))}{q_j(x^0, y)} = \sum_{j \in I(x^0)} \mu_j \nabla g_j(x^0)^T \eta(y, x^0) \\
= \sum_{i=1}^{k} \lambda_i \nabla f_i(x^0)^T \eta(y, x^0) = \sum_{i=1}^{k} \frac{\lambda_i (f_i(y) - f_i(x^0))}{p_i(x^0, y)} > 0,
\]

which is a contradiction.

Conversely, suppose that $x^0$ is an efficient solution for (P). For $1 \leq r \leq k$, the system
\[
\begin{align*}
\nabla g_j(x^0)^T \eta(x, x^0) & \leq 0 \quad j \in I(x^0) \\
\nabla f_i(x^0)^T \eta(x, x^0) & \geq 0 \quad i = 1, 2, \ldots, r-1, r+1, \ldots, k \\
\n\nabla f_r(x^0)^T \eta(x, x^0) & > 0
\end{align*}
\]

(3)

has no solution \( x \in X \). Suppose there exist \( y \) such that

\[
\begin{align*}
\nabla g_j(x^0)^T \eta(y, x^0) & \leq 0 \quad j \in I(x^0) \\
\nabla f_i(x^0)^T \eta(y, x^0) & \geq 0 \quad i \neq r \\
\n\nabla f_r(x^0)^T \eta(y, x^0) & > 0.
\end{align*}
\]

Then \( g_j(y) \leq g_j(x^0), f_i(y) \geq f_i(x^0), i \neq r \) and \( f_r(y) \geq f_r(x^0) \) but \( f_r(y) \neq f_r(x^0) \) since \( f_r(y) = f_r(x^0) \) if and only if \( \nabla f_r(x^0)^T \eta(y, x^0) = 0 \). Therefore \( f_r(y) > f_r(x^0) \), which contradicts that \( x^0 \) is an efficient solution. It follows that (3) has no solution. By Farkas' lemma (Mangasarian [6]), there exist \( \lambda_j \geq 0 \) and \( \mu_r \geq 0 \) such that

\[
\sum_{i \neq r} \lambda_i \nabla f_i(x^0) + \nabla f_r(x^0) = \sum_{j \geq 1} \mu_j \nabla g_j(x^0).
\]

Summing over \( r \) we get (1) with \( \lambda_i = 1 + \sum_{i \neq r} \lambda_i \) and \( \mu_j = \sum_{i \neq r} \mu_j \).

Chew and Choo [2] proved that efficiency and properly efficiency are equivalent under certain conditions for a particular case of problem (P) when the functions involved are pseudolinear. We are going to show that the same is true for \( \eta \)-pseudolinear functions.

**Definition 4.** (Chew and Choo [2]). A feasible point \( x^0 \) is said to satisfy the **boundedness condition** if the set

\[
\left\{ \frac{p_i(x^0, x)}{p_j(x^0, x)} \mid x \in X, f_i(x^0) < f_j(x), f_j(x^0) > f_j(x), 1 \leq i, j \leq k \right\}
\]

(4)

is bounded from above.

**Proposition 2.** Assume the same hypotheses as in Proposition 1. Then every efficient solution of (P) that satisfies the boundedness condition is properly efficient.

**Proof:** Let \( x^0 \) be an efficient solution of (P). Then it follows from Proposition 1 that there exist \( \lambda_j > 0 \) and \( \mu_j \geq 0 \) such that \( \sum_{i=1}^k \lambda_i \nabla f_i(x^0) = \sum_{j \geq 1} \mu_j \nabla g_j(x^0) \). Therefore for any feasible \( x \),

\[
\sum_{i=1}^k \lambda_i \nabla f_i(x^0)^T \eta(x, x^0) = \sum_{j \geq 1} \mu_j \nabla g_j(x^0)^T \eta(x, x^0).
\]

Notice that

\[
\sum_{i=1}^k \lambda_i \nabla f_i(x^0)^T \eta(x, x^0) \leq 0.
\]

(5)

Otherwise, we would obtain a contradiction as in (2), Proposition 1.
Since the set defined by (4) is bounded from above, the following set is also bounded from above:

\[
\left\{ (k-1) \frac{\lambda_i p_i(x^0, x)}{\lambda_j p_j(x^0, x)} \mid x \in X, f_i(x^0) < f_i(x), f_j(x^0) > f_j(x), 1 \leq i, j \leq k \right\}.
\]

(6)

Let \( M \) be a positive real number that is an upper bound of the set defined by (6). We are going to show that \( x^0 \) is properly efficient.

Suppose that there exist \( r \) and \( x \in X \) such that \( f_r(x) > f_r(x^0) \). Then

\[
\nabla f_r(x^0)^{\top} \eta(x, x^0) > 0.
\]

(7)

Let

\[
-\lambda \nabla f_r(x^0)^{\top} \eta(x, x^0) = \max \left\{ -\lambda \nabla f_r(x^0)^{\top} \eta(x, x^0) \mid \nabla f_r(x^0)^{\top} \eta(x, x^0) < 0 \right\}.
\]

(8)

From (5), (7), and (8), we obtain

\[
\lambda \nabla f_r(x^0)^{\top} \eta(x, x^0) \leq (k-1)(-\lambda \nabla f_r(x^0)^{\top} \eta(x, x^0)).
\]

Therefore

\[
f_r(x) - f_r(x^0) \leq (k-1) \frac{\lambda_i p_i(x^0, x)}{\lambda_j p_j(x^0, x)} (f_j(x^0) - f_j(x)),
\]

and given the choice of \( M \),

\[
f_r(x) - f_r(x^0) \leq M (f_j(x^0) - f_j(x)).
\]

Thus, \( x^0 \) is properly efficient.

References